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# Group 5

# *Investment decisions under uncertainty*

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# Contents

**Abstract.** In the present work, we develop a solution for a firm which is uncertain about future revenues. The firm has the option to invest in a new product or technology, suspend production or even exit the market, according to its expectations about the future. We study the opportunities when the firm should investigate. As soon as we make an investment, our profit is $\pi(.)$. We solve the problem by using stochastic differential equations, Itô's formula and optimization algorithms.

## 5.1 Introduction

The problem that we have worked on consists of finding the optimal investment strategy when a firm wants to change markets. We have first derived an investment threshold of the demand and later we have worked out the time distribution to get into that threshold. Different models for the demand and benefits have been considered in order to approach the problem from a more and more realistic way.

## 5.2 General model

Let us consider a firm which is currently producing one product. It has the opportunity to make an investment or change to a more risky market. As soon as the firm makes an investment, it has an investment cost $h(x)$ and a running payoff $\Pi(x)$. The payoff depends on a stochastic process $X = \{X(\tau), \tau \geq 0\}$[1], where $\tau \in T$ is the investment time in an infinite set of stopping times. In order to avoid arbitrage, we must have

$$\int_0^\infty e^{-Rs}\mathbf{E}[\Pi(X(s)|X(0) = x]ds < \infty.$$

where $R$ is the interest rate and is assumed to be a constant. Of course, if it is the case were the latter integral does not converge, we should invest at time 0, which would imply an arbitradge. The value of our option if we invest at a certain time $\tau$ satisfies

$$f(x,\tau) = \mathbf{E}[\int_\tau^\infty e^{-Rs}\Pi(X(s))ds - e^{-R\tau}h(X(\tau))|X(0) = x], \tag{5.1}$$

We wish to find the value function $V(x)$, which gives us the maximum value of the option. In this way, $V(x)$ satisfies

$$V(x) = \underset{\tau\in T}{sup}\ f(x,\tau). \tag{5.2}$$

Of course, the optimal $\tau$ is not going a fixed time. The time $\tau$ depends in the actual values of the stochastic process. Due to the stochasticity of the problem,

[1] This stochastic process can be understood as the demand, but in the most general case it could be another magnitude.

one can have that at a certain time $\tau$ the value of $X$ is really high or really low. For this reason, we will seek at which value of $X$ shall we do the investment, which will give a random distribution for the stopping time $\tau$. The process $X(s)$ satisfies a stochastic differential equation (SDE)

$$dX(s) = \tilde{\mu}(s, X(s))ds + \tilde{\sigma}(s, X(s))dW(s), \tag{5.3}$$

where $W(s)$ is a Brownian Motion, $\tilde{\mu}(s, X(s))$ is the drift and $\tilde{\sigma}(s, X(s))$ is the volatility. Since $W(s)$ is nowhere differentiable, we need to apply the Itô's formula to integrate the SDE. The Itô's formula is given by

$$\begin{aligned} df(t, x(t)) &= \left(\frac{\partial f}{\partial t} + \tilde{\mu}(t, x(t))\frac{\partial f}{\partial x} + \frac{1}{2}\tilde{\sigma}^2(t, x(t))\frac{\partial^2 f}{\partial x^2}\right)dt \\ &\quad + \tilde{\sigma}(t, x(t))\frac{\partial f}{\partial x}dW(t). \end{aligned}$$

For a Geometric Brownian motion $X$ with a class of finite stopping times $T$, the drift and volatility are given by $\tilde{\mu}(\tau, x) = \mu x$ and $\tilde{\sigma}(\tau, x) = \sigma x$, where $\mu$ and $\sigma$ are constants. Then Equation (5.3) can be written as

$$\frac{dX(s)}{X(s)} = \mu ds + \sigma dW(s). \tag{5.4}$$

We apply Itô's formula to $f$ in Equation (5.1) and finally, using stochastic analysis techniques one can check that $V$ satisfies an ordinary differential equation

$$\Pi(y) + \tilde{\mu}V''(y) + \frac{1}{2}\tilde{\sigma}^2 V\text{"}(y) - RV(y) = 0. \tag{5.5}$$

The function $V(y)$ must also satisfy $V(y) = h(y)$ and $V'(y) = h'(y)$ due to optimality. If this conditions are not fulfilled, one can always find an slightly better solution to the optimization problem 5.2. Through this derivation, one can simpilfy his stochastic optimization process into a differential equation free-boundary problem. In the cases that we will discuss, this problem can be solved analitically, which is a great advantadge since we do not need to solve a free-boundary problem numerically.

## 5.3 Case I - The simplest linear model

### 5.3.1 Investment threshold

Suppose we have a product whose price satisfies a Geometric Brownian motion $X$. We set $\Pi(x) = ax$ and $h(x) = I$, where $a$ and $I$ are constants. Then according to Equation (5.5), $V$ satisfies

$$ay + \mu y V'(y) + \frac{1}{2}\sigma^2 y^2 V''(y) - RV(y) = 0, \tag{5.6}$$

with boundary condition at the optimal point $y^*$

$$V(y^*) = h(y^*) = I, \tag{5.7}$$
$$V'(y^*) = h'(y^*) = 0. \tag{5.8}$$

To avoid an arbitrage, we also need to have

$$\int_0^\infty \mathbf{E}[x(s)]e^{-Rs} < \infty. \tag{5.9}$$

Therefore, we have

$$\int_0^\infty e^{(\mu-R)s} < 0. \tag{5.10}$$

Hence, we need $R > \mu$. Solving Equation (5.6) with boundary conditions (5.7) - (5.8), we derive the solution for $V$ and the investment threshold $y^*$

$$V = Cy^{r^*} + \frac{a}{R-\mu}y, \tag{5.11}$$

$$y^* = = \frac{I(R-\mu)}{a(1-1/r^*)}, \tag{5.12}$$

where $r^*$ and $C$ are given by

$$r^* = \frac{\sqrt{(\mu - 1/2\ \sigma^2)^2 + 2\sigma^2 R} - (\mu - 1/2\ \sigma^2)}{\sigma^2} \tag{5.13}$$

$$C = \frac{a}{r^*(\mu-R)}y^{*1-r^*}. \tag{5.14}$$

### 5.3.2 Comparative statics

From the company viewpoint it is interesting to see how does the investment threshold changes with $\mu$ and $\sigma$, as in real life one does not have constant $\mu$ or $\sigma$. For this reason, one can adjust slightly his investment threshold when one sees an increase or decrease of the drift. Therefore, we wonder how the investment threshold $y^*$ changes when we have different drift $\mu$ and volatility $\sigma$. In order to see this, we look at $\partial y/\partial\mu$ and $\partial y/\partial\sigma$. Using chain rule, we have

$$\frac{\partial y^*}{\partial\sigma} = \frac{\partial y^*}{\partial r^*}\frac{\partial r^*}{\partial\sigma^2}\frac{\partial\sigma^2}{\partial\sigma}. \tag{5.15}$$

As $r^*$ and $\sigma^2$ satisfies $\mu r^* + 1/2\sigma^2(r^{*2} - r^*) - R = 0$ from solving the Cauchy-Euler Equation (5.6), we can use the Implicit Function Theorem to compute $\partial r^*/\partial\sigma^2$. Hence, we have

$$\frac{\partial y^*}{\partial\sigma} = -\frac{Ir^*(R-\mu)}{a(r^*-1)}\frac{1}{\mu(r^*-1/2)\sigma} \tag{5.16}$$

We find out that $\partial y^*/\partial\sigma$ is always positive but with $\partial y^*/\partial\mu$ is more complicated. After several algebraic computations we find that $\partial y^*/\partial\mu$ is positive when $R^2 + \mu^2 - R + 4R\mu^2/\sigma^2 - 2\mu R > 0$. We also find that

$$\frac{\partial r^*}{\partial\mu} = -1/\sigma^2 + 1/\sigma^2\frac{\mu - 1/2\sigma^2}{\sqrt{(\mu-1/2\sigma^2)^2 + 2R\sigma^2}} \tag{5.17}$$

should be negative. Our $r^*$ is decreasing with different values of $\mu$ .

### 5.3.3 Time simulations

In the previous sections, we have obtained the investment threshold for our first simple case. In this section we wonder how long does it take to get to that threshold. Due to the stochasticity of the problem, one has that the time that it takes to get to the threshold is actually a random variable. We have run numerical simulations in order to see which is the distribution of the time that it takes to get the investment threshold value. This is shown in the following figures.

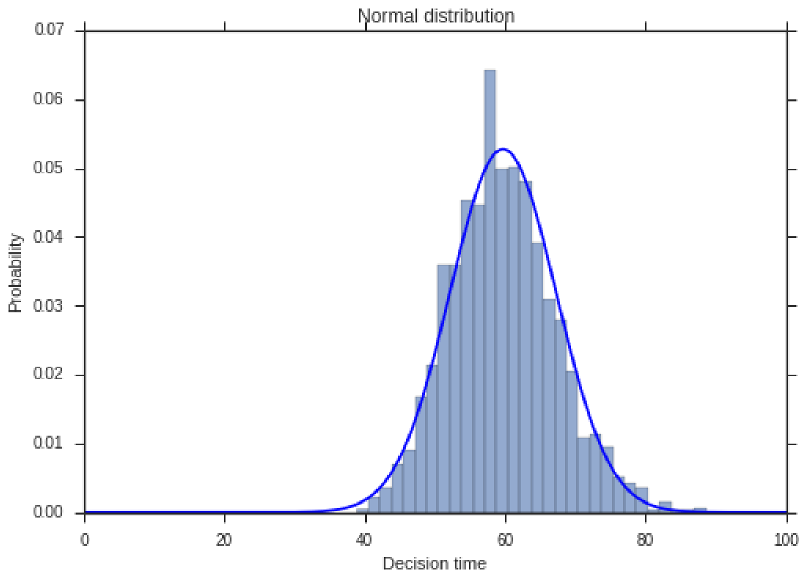


Figure 5.1: Time simulations

Kolgomorov-Smirnov test rejects normality for every set of simulations.

In order to make it a little bit less trivial, one can have that some of the simulations are contaminated and have an slightly different drift. This would model uncertainty on the values of the parameters. We have evaluated how does the mean and the variance of our stopping time distributions as a function of the number of the noisy runs and the change of the drift. One can see that the mean of the time distribution is correlated with the number of noisy runs, while the standard deviation is independent.

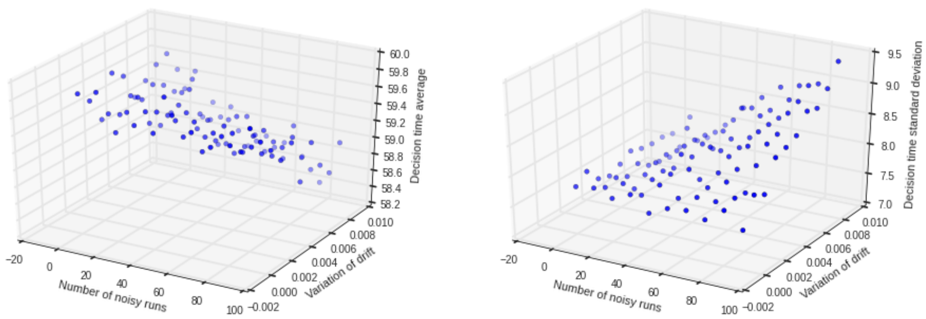


Figure 5.2: Noisy time simulations

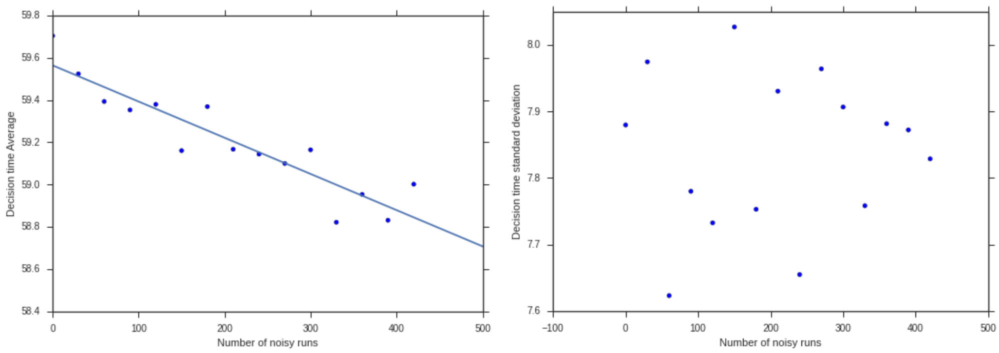


Figure 5.3: Variation of drift = 10%, Total number of runs = 2500

## 5.4 Case II - $\Pi(y) = ay^\theta$

We can change our problem to a more general profit function.
Now we assume our profit function is $\Pi(y) = ay^\theta$, where $\theta > 0$. We substitute $\Pi(y)$ into Equation (5.5) and we get

$$ay^\theta + \mu y V'(y) + \frac{1}{2}\sigma^2 y^2 V''(y) - RV(y) = 0, \tag{5.18}$$

with boundary conditions (5.7) - (5.8) at the threshold $y^*$. Solving the ODE we then have

$$V = Ay^{r^*} + Cy^\theta, \tag{5.19}$$

$$y^* = \sqrt[\theta]{\frac{Ir^*}{C(r^* - \theta)}}, \tag{5.20}$$

where $A$, $C$ and $r^*$ are given by

$$C = -\frac{a}{\mu\theta + 1/2\sigma^2\theta(\theta-1) - R}, \tag{5.21}$$

$$r^* = \frac{\sqrt{(\mu - 1/2\ \sigma^2)^2 + 2\sigma^2 R} - (\mu - 1/2\ \sigma^2)}{\sigma^2} \tag{5.22}$$

$$A = -\frac{C\theta}{r^*} y^{*\theta - r^*}. \tag{5.23}$$

## 5.5 Case III - Investing in two products

Let us consider a more realistic scenario that we have two products to invest. We are currently in the market of Product 1. At some point $t$ we can change to invest Product 2, which is more risky. The profit functions for two products are given as

$$\Pi_1(x) = a_1 x - b_1, \tag{5.24}$$

$$\Pi_2(x) = a_2 x - b_2, \tag{5.25}$$

where we set $a_2 > a_1 > 0$ and $b_2 > b_1 > 0$ to avoid an arbitrage. We assume there is no cost when switching from one market to the other. The value function $V(x)$ in this case is

$$V(x) = \sup_{\tau \in T} \mathbf{E}[\int_0^\tau e^{-Rs}\Pi_1(X(s))ds + \int_\tau^\infty e^{-Rs}\Pi_2(X(s))ds]. \tag{5.26}$$

Rearranging Equation (5.26), we have

$$V(x) = \sup_{\tau \in T} \mathbf{E}[\int_0^\infty e^{-Rs}\Pi_1(X(s))ds + \int_\tau^\infty e^{-Rs}\big(\Pi_2(X(s)) - \Pi_1(X(s))\big)ds]. \tag{5.27}$$

We set the new profit function $\tilde{\Pi}(x)$ to be $\big(\Pi_2(x(s)) - \Pi_1(x(s))\big)$. Since the first integral of Equation (5.27) is a constant, optimising Equation (5.27) is equivalent to optimising

$$V(x) = \sup_{\tau \in T} \mathbf{E}[\int_\tau^\infty e^{-Rs}\tilde{\Pi}(X(s))ds]. \tag{5.28}$$

Following the discussion in Section 5.3.1, $V$ still satisfies Equation (5.5), which gives

$$cy - d + \mu y V'(y) + \frac{1}{2}\sigma^2 y^2 V''(y) - RV(y) = 0, \tag{5.29}$$

where $c = a_2 - a_1$ and $d = b_2 - b_1$. We solve Equation (5.29) subject to boundary conditions (5.7) - (5.8) at the threshold $y^*$ with $I = 0$. Then we have

$$V = Ay^{r^*} + \frac{c}{R-\mu}y - \frac{b_1}{R}, \tag{5.30}$$

$$y^* = \frac{b_1 r^*(R-\mu)}{Rc(r^*-1)}, \tag{5.31}$$

where $A$ and $r^*$ are given by

$$A = \frac{c}{R-\mu}, \tag{5.32}$$

$$r^* = \frac{\sqrt{(\mu - 1/2\ \sigma^2)^2 + 2\sigma^2 R} - (\mu - 1/2\ \sigma^2)}{\sigma^2}. \tag{5.33}$$

We wonder how the change of two profit functions will affect our optimal point. To look at this, we first fix $\Pi_1$ and vary $\Pi_2$. Meanwhile, we also fix their intersection point so that the only factor that affects the location of the optimal point is the level of risk of Product 2, which can be interpreted as the gradient of $\Pi_2$. The intersection of $\Pi_1$ and $\Pi_2$, $\hat{y}$ is given by $d/c$. We compare $\hat{y}$ with $y^*$ and we find out that only the value of $b_2$ will affect the location of $y^*$. In general, $y^*$ increases when $b_2$ increases.

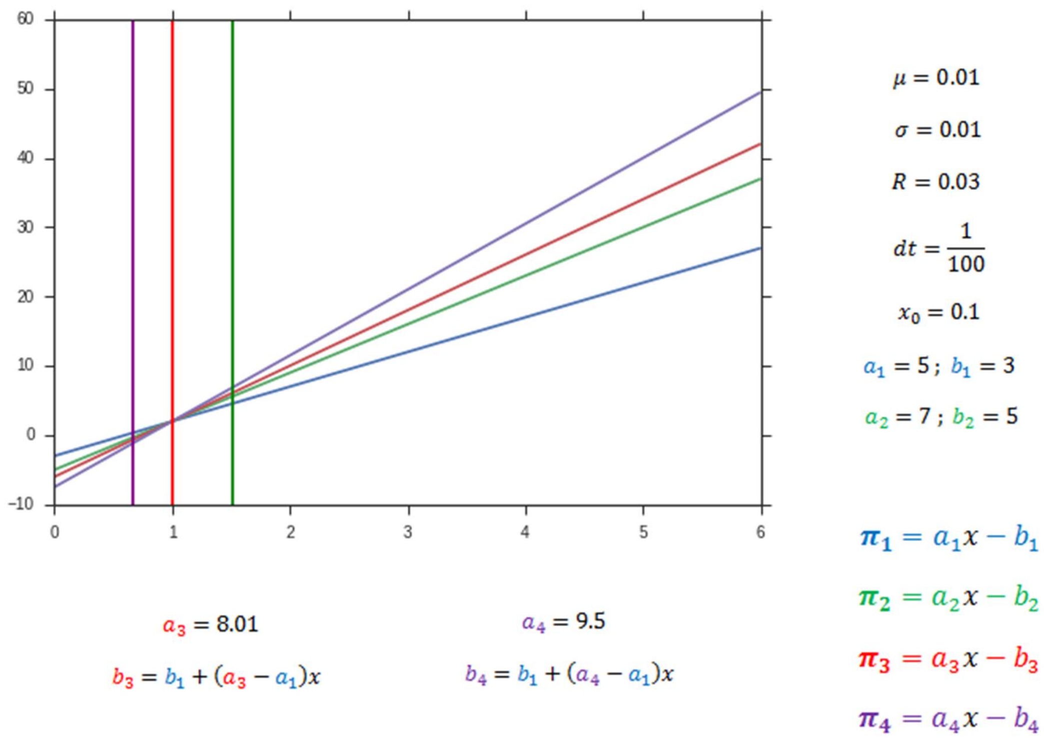


Figure 5.4: Angle illustration

## 5.6 Case IV - Jump model

In real world business the market price may experience a sudden jump. These jumps are usually related to political changes or natural catastrophes that can happen at any time and produce huge changes in the economy. The most simple model to model these jumps is a jump-diffusion process.

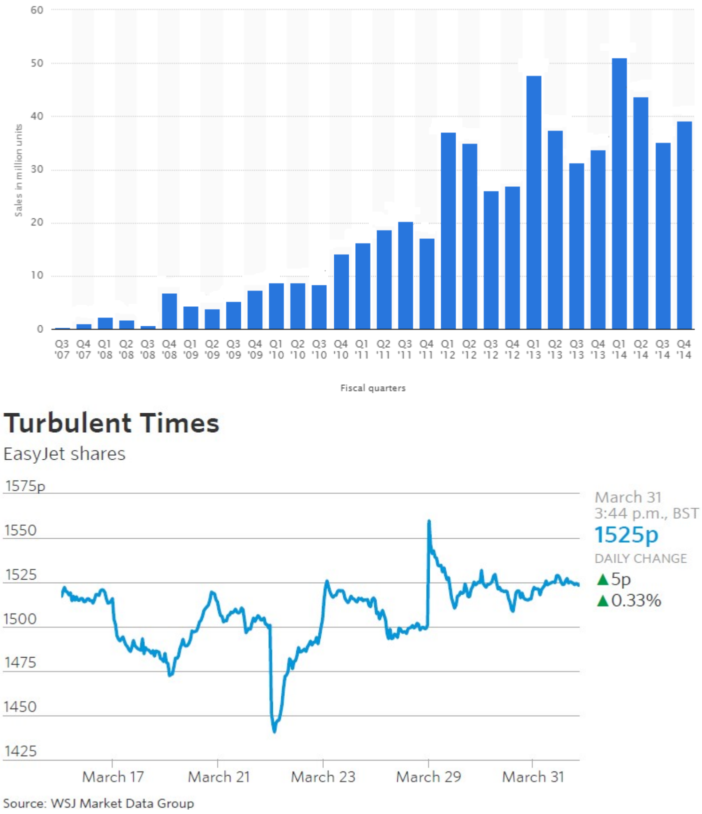


Figure 5.5: Those graphs show us the possitve and negative jumps that happen in market price.

We use a jump-difussion process $X(t)$, which considers fixed-height jumps and the time between jumps as an exponential distribution. $X(t)$ satisfies a stochastic differential equation that

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t) + X(t)udN(t), \tag{5.34}$$

where $u$ is the jump size and $N(t)$ is the Poisson process with parameter $\lambda$. By applying Itô's formula, we get an SDE for the price function

$$\Pi(y) + \mu y V'(y) + \frac{1}{2}\sigma^2 y^2 V''(y) - RV(y) + \lambda\big(V(x(1+u)) - V(x)\big) = 0, \tag{5.35}$$

with boundary conditions (5.7) - (5.8) at the threshold $y^*$. We set $\Pi(y) = ay$.

Then the solutions for $V$ and $y^*$ are given by

$$V = Ay^{r^*} + \frac{a}{R-\mu-\lambda u}y, \tag{5.36}$$

$$y^* = \frac{I(R-\mu-\lambda u)}{a(1-1/r^*)}, \tag{5.37}$$

where $A$ is given by

$$A = \frac{a}{r^*(\lambda u+\mu-R)}y^{*1-r^*}, \tag{5.38}$$

$$f(r) = \mu r + \frac{1}{2}\sigma^2 r(r-1) - R + \lambda\big((1+u)^r - 1\big) \tag{5.39}$$

and $r^*$ satisfies

$$f(r^*) = 0 \tag{5.40}$$

### 5.6.1 Comperative statics

We wonder how $y^*$ changes with respect to $\mu$, $\sigma$ and $u$. To see this, we take the partial derivative with respect to each parameter and apply the Implicit Function Theorem. It turns out that the sign of these derivatives is the same as $\partial f/\partial\mu$, $\partial f/\partial\sigma$ and $\partial f/\partial u$ where

$$\frac{\partial f}{\partial \mu} = r, \tag{5.41}$$

$$\frac{\partial f}{\partial \sigma} = r(r-1), \tag{5.42}$$

$$\frac{\partial f}{\partial u} = \lambda r(r-1). \tag{5.43}$$

Using the program Wolfram Mathematica we numerically find out that all first derivatives increase.

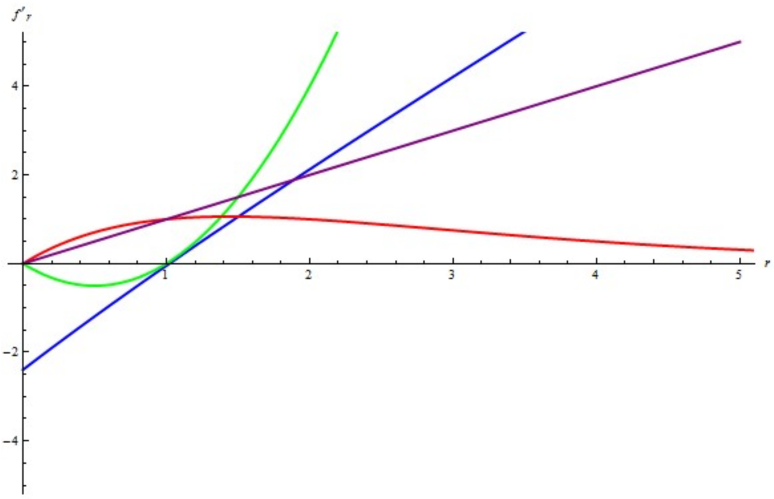


Figure 5.6: This figure shows that all first derivates increase.Where the green graphic is for $\partial f/\partial\sigma$, the red one - $\partial f/\partial u$, the blue one - $\partial f/\partial r$ and the purple - $\partial f/\partial\mu$.

We find out that $f(r)$ increases when increase each parameter, respectively.

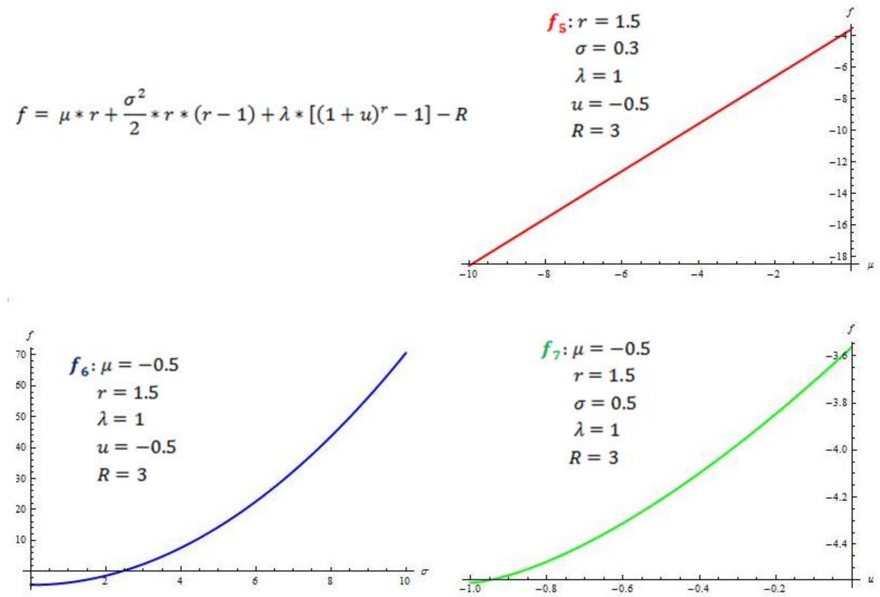


Figure 5.7: Numerical analysis of the growth of the $f(r)$

To ensure the monotonicity $y^* > 0$, we need to have $r > 1$ and therefore $f(r)$ is an increasing function.

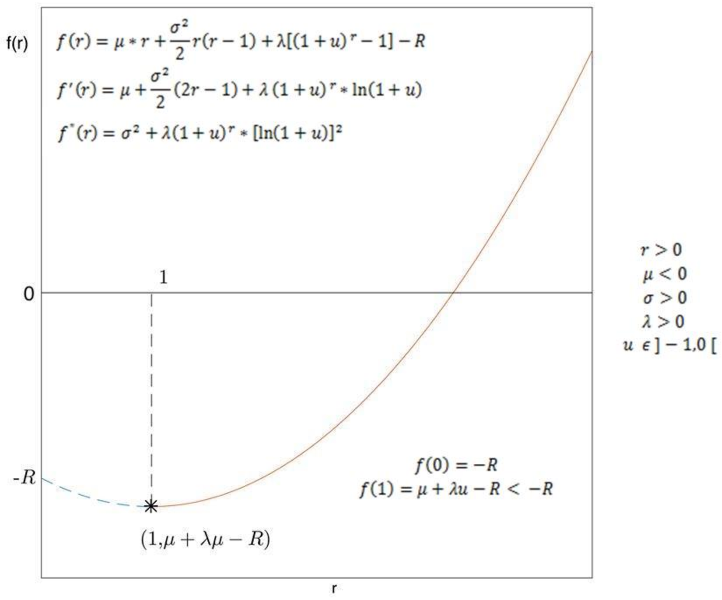


Figure 5.8: The function is monotonically increasing when $r > 1$

### 5.6.2 Random distribution of the stopping time

Again, we can compute the random distribution of the stopping time by simulation. In the next Figure, we can see that the Gamma distribution can do a proper fit for this random variable.

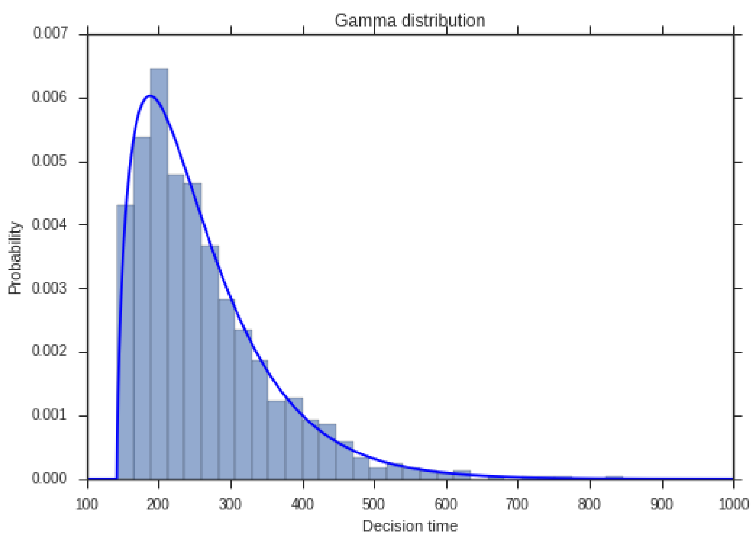


Figure 5.9: Time simulation

$\chi^2$ test does not reject when asymptotic hypothesis of the test are fulfilled, but it is too sensitive to the size of the bins. For this reason, one may want to look for a more robust process.

## 5.7 Case V - Quantity dependence

In real life when we consider entering two markets the value function may not only depend on the demand, but also the investment quantity of each market. Let us consider a profit function for product i

$$\Pi_i(x) = xk_i(1 - n_ik_i)\text{for i= 1, 2} \tag{5.44}$$

where $k_i$ is the quantity produced of product i and $n_i$ is a constant related to the dependence of the prize on the offer. In this way, the value function satisfes

$$V(x) = \underset{\tau \in T}{sup}\, \mathbf{E}[\int_0^\tau e^{-Rs}\Pi_1(X(s))ds + \int_\tau^\infty e^{-Rs}\Pi_2(X(s))ds - \delta k_2 e^{-R\tau}|X(0) = x] \tag{5.45}$$

where $\delta$ is the cost of changing markets per production unit. We set $\tilde{\Pi}(x)$ to be $\Pi(x) + R\delta k_2$. Rearranging $V$ we get

$$V(x) = \underset{\tau \in T}{sup}\, \mathbf{E}[\int_0^\tau e^{-Rs}(\Pi_1(X(s)) + R\delta k_2)ds + \int_\tau^\infty e^{-Rs}\Pi_2(X(s))ds] \tag{5.46}$$

which is equivalent to the frst changing markets scenario that has been already solved. Finally, we just have to take the value function V(x; $k_2$) and optimize it with respect to $k_2$ for every x.

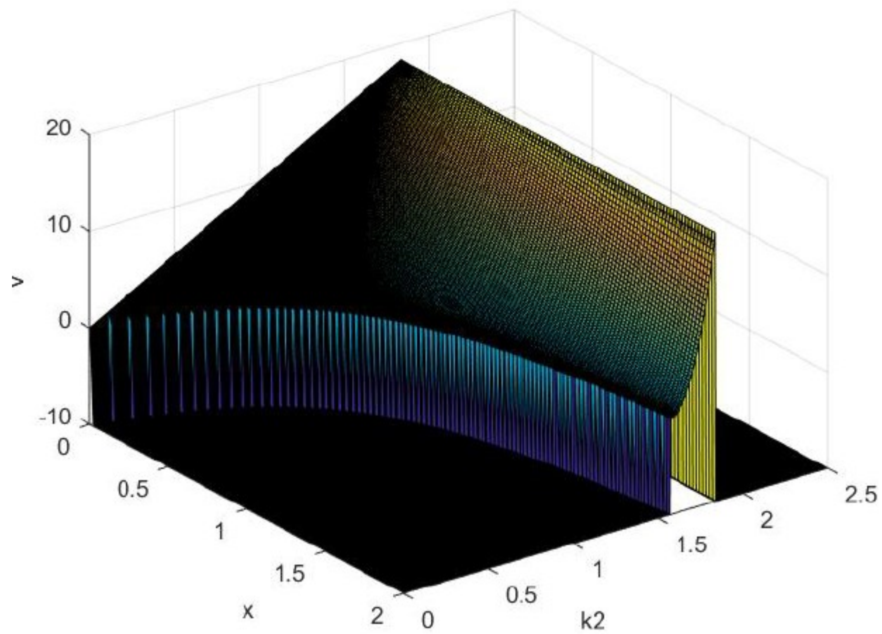


Figure 5.10: Optimal $k_2$ as a function of $x$

One can see that the optimal $k_2$ is actually independent of $x$.

## 5.8 Conclusion

The main conclusions of our work are the following ones:

- Analytics is necessary, as we are dealing with a free boundary problem.
- Complex cases can be transformed into simpler cases with algebraic tricks.

While doing approximations, we have realized that there is lots of future work to be done: we can consider positive/random jumps, we can try to work out the estimation of parameters or we can work as if we had a finite horizon to invest.

# Bibliography

[1] Hagspiel, Verena, Kuno JM Huisman, and Cludia Nunes, "Optimal technology adoption when the arrival rate of new technologies changes." European Journal of Operational Research 243.3 (2015):pp. 897-911.

[2] Hagspiel, Verena, Kuno JM Huisman, Kort, Peter and Cludia Nunes. "How to escape a declining market: capacity investment or exit." To be published in European Journal of Operational Research.

[3] Kevin Ross, Stochastic Control in Continuous Time. **http://www.swarthmore.edu/NatSci/kross1/Stat220notes.pdf**

[4] Pauli Murto, Timing of investment under technological and revenue-related uncertainties.Journal of Economic Dynamics and Control Volume 31, Issue 5, 2007, pp. 1473 − 1497 .

[5] A. Dixit and R. Pindyck, Investment under uncertainty, chapters: 4, 5 and 6, Princeton University Press, 1994.

[6] Huisman, Kuno J.M., Technology Investment: A Game theoretical game approach, chapters:1, 2 and 3, Springer USA, 2001.