Volume preservation 3D segmentation

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- Input (gray-scale) image: $\bar{u} : \Omega \to [0, \nu]$.
  $\Omega \to n_1 \times n_2 \times n_3$ discrete grid, $\nu \in \mathbb{Z} \to$ maximal intensity.
  $\bar{u} \in [0, \nu]^{\mathbf{n}} \to$ reshaped column-wise into vector form.
- Index set: $\mathbb{I}_{\mathbf{n}} := \{1, \ldots, \mathbf{n}\} = L_0 \cup L_1 \cup U$
  $L_0 = \{1, \ldots, \ell\}$, $L_1 = \{\ell + 1, \ldots, 2\ell\}$.
- Segment membership vector v:

$$v(i) = \begin{cases} 0, & \text{i-th voxel is background ("air")}, \\ 1, & \text{i-th voxel is foreground ("material")}. \end{cases}$$

- Goal: Determine $v|_U$, knowing that $v|_{L_0} = 0,\ v|_{L_1} = 1$.

Direct threshold segmentation $v(i) = \left\lfloor 1 + \frac{\bar{u}(i) - c}{\nu} \right\rfloor$ works poor in the presence of noise, blur or other artifacts in $\bar{u}$. For good results, *more advanced methods* are needed.

Supervised 2-Laplacian segmentation [Kang, Shafei, Steidl 2014]

$$\operatorname*{argmin}_{v\in\{0,1\}^{\mathbf{n}}} \frac{1}{2}\sum_{i,j=1}^{\mathbf{n}} w_{i,j}\big(v(i)-v(j)\big)^2 \quad \text{subject to} \quad v_L(i) = \begin{cases} 0, & i \in L_0, \\ 1, & i \in L_1. \end{cases}$$

Weights $w_{i,j}$ are defined via:

$$w_{i,j}^{geo} := \begin{cases} 1/6 & \text{if } \mathcal{N}_i^{geo}, \\ 0 & \text{otherwise}, \end{cases} \qquad w_{i,j}^{pho,lab} := \begin{cases} c_i e^{-\|F(i)-F(j)\|_2^2} & \text{if } j \in \mathcal{N}_i^{pho} \cup \mathcal{N}_i^{lab}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{N}_i^{geo} = \{j : \|j-i\|_1 = 1\}; \qquad |\mathcal{N}_i^{pho}| = 4; \qquad \mathcal{N}_i^{lab} = L_0 \cup L_1;$$

$$F(i) = \frac{1}{12}\left(6\bar{u}(i) + \sum_{j\in\mathcal{N}_i^{geo}} \bar{u}(j)\right).$$

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Weights $w_{i,j}$ are defined via:

$$\begin{aligned} W^* &= \frac{1}{1+\nu^{\text{pho}}} W^{geo} + \frac{\nu^{\text{pho}}}{1+\nu^{\text{pho}}} W^{pho} \\ \tilde{W} &= \max\left\{ \frac{\nu^{\text{lab}}}{1+\nu^{\text{lab}}} W^{lab}, \frac{1}{1+\nu^{\text{lab}}} W^* \right\} \\ W &= \max\left\{ \tilde{W}, \tilde{W}^T \right\} = \begin{pmatrix} W_{LL} & W_{LU} \\ W_{UL} & W_{UU} \end{pmatrix}. \end{aligned}$$

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The minimizer is given via

$$\underbrace{(D_{UU} - W_{UU})}_{Q}\,\bar{v}_U = \underbrace{W_{UL} v_L}_{q}.$$

Q – sparse, M-matrix, q – nonnegative $\Rightarrow$ use parallelized CG for solving.

Such segmentation methods work fine for well-separated, smooth phases, but their performance is unclear in the presence of big interaction.

Constrained segmentation

In (homogeneous) porous media, the "air" consists of multiple, non-structured, possibly not even connected pores of various size and shape, that "cut" through the material. Combined with the inevitable noise and blur the input image possesses, unconstrained segmentation is often poor and unreliable.

To improve the result, we incorporate some physical properties of the scanned specimen in the segmentation process as constraints. In particular, the volume of the solid phase can be determined from the material's density and weight measurements and can be a priori prescribed (**mass conservation**).

$$\operatorname*{argmin}_{v_U \in \{0,1\}^{n_1}} \langle Q v_U, v_U \rangle \quad \text{subject to} \quad \|v_U\|_0 = N_1. \tag{1}$$

The ℓ_0 pseudo-norm is non-convex $\Rightarrow$ the problem is NP-Hard.

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The $\{0,1\}^{\mathbf{n}_1}$ set is discrete $\Rightarrow$ the problem needs to be relaxed.

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$$\operatorname*{argmin}_{v_U \in [0,1]^{n_1}} \langle Q v_U, v_U \rangle \quad \text{subject to} \quad e^T v_U = N_1. \tag{1}$$

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Therefore further relaxations are needed!

Constraint Segmentation: Proposed Algorithms

Algorithm 1

$$\operatorname*{argmin}_{v_U,\lambda} \frac{1}{2} v_U^T Q v_U - q^T v_U + \lambda(e^T v_U - N_1) - \mu(v_U^T v_U - N_1). \tag{2}$$

λ – Lagrange multiplier; μ – nonnegative penalizer.

The solution is given by

$$\begin{pmatrix} Q - 2\mu I & e \\ e^T & 0 \end{pmatrix} \begin{pmatrix} \bar{v}_U \\ \lambda \end{pmatrix} = \begin{pmatrix} q \\ N_1 \end{pmatrix}. \tag{3}$$

We solve the system via Schur complement and parallelized CG. Then, we take the N_1 pixels of highest value of $\bar{v}_U$ to be foreground and set the rest to the background.

Constraint Segmentation: Proposed Algorithms

Algorithm 2

$$\operatorname*{argmin}_{0 \le v_U \le 1} \|\bar{Q} v_U\|_2^2 \quad \text{s.t.} \quad e^T v_U \ge N_1,$$

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Algorithm 2

$$\underset{v_U \in \mathbb{R}^{\mathbf{n}_1}, x \in \mathbb{R}^{3\mathbf{n}_1}}{\operatorname{argmin}} \left\{ \langle 0, v_U \rangle + \iota_H(x_1) + \|x_2\|_2^2 + \iota_{[0,1]^{\mathbf{n}_1}}(x_3) \right\} \text{ s.t. } \begin{pmatrix} I \\ \bar{Q} \\ I \end{pmatrix} v_U = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

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Initialization: $q_{1,2,3}^{(0)} = 0$, $x_{1,3}^{(0)} = e$, $x_2^{(0)} = \bar{Q}e$, $\gamma \in (0,1)$.
For $k = 0, 1, \dots$ repeat until a stopping criterion is reached

1. $v_U^{(k+1)} = (\bar{Q}^T\bar{Q} + 2I)^{-1}\left(\left(x_1^{(k)} - q_1^{(k)}\right) + \bar{Q}^T\left(x_2^{(k)} - q_2^{(k)}\right) + \left(x_3^{(k)} - q_3^{(k)}\right)\right)$
2. $x_1^{(k+1)} = \begin{cases} q_1^{(k)} + v_U^{(k+1)}, & \left(q_1^{(k)} + v_U^{(k+1)}\right) \in H, \\ q_1^{(k)} + v_U^{(k+1)} + \frac{N_1 - e^T(q_1^{(k)} + v_U^{(k+1)})}{\mathbf{n}_1^2} e, & \text{otherwise.} \end{cases}$
3. $x_2^{(k+1)} = \gamma\left(q_2^{(k)} + \bar{Q}v_U^{(k+1)}\right)$
4. $x_3^{(k+1)} = \min\left(1, \max\left(0, q_3^{(k)} + v_U^{(k+1)}\right)\right)$
5. $q_i^{(k+1)} = q_i^{(k)} + v_U^{(k+1)} - x_i^{(k+1)}$, $i = 1, 3$, $\qquad q_2^{(k+1)} = q_2^{(k)} + \bar{Q}v_U^{(k+1)} - x_2^{(k+1)}$.

Reconstruction of noisy and blurry bone

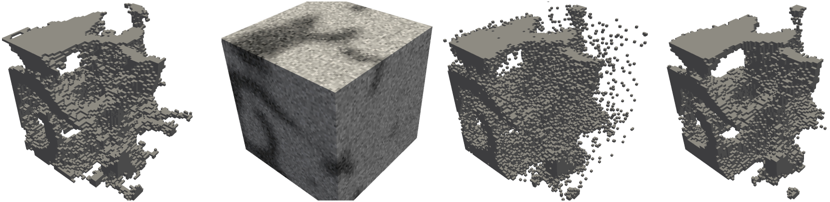

Figure : From left to right: Original (segmented) bone, blurred and noisy input image, direct constrained segmentation, Algorithm 2 segmentation.

hausd \ #voxels	original	direct	Alg. 1 $\mu = 0$	Alg. 1 $\mu > 0$	Alg. 2
original	$*$	21796	16048	16156	15524
direct	25547	$*$	10476	10412	11346
Alg. 1 $\mu = 0$	19417	14218	$*$	278	1486
Alg. 1 $\mu > 0$	19601	14182	289	$*$	1722
Alg. 2	18857	15140	1509	1764	$*$

Table : Segmentation comparison: #Pixel difference (above the diagonal) and Hausdorff distance (below the diagonal) between different methods.

Sphere expansion

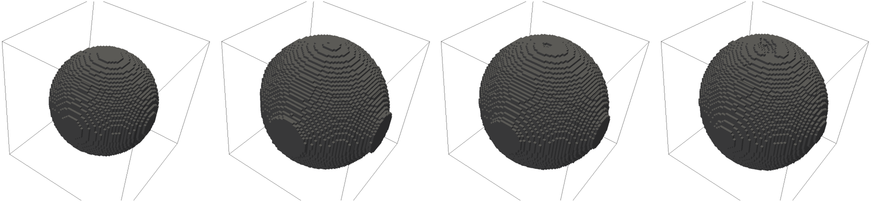

Figure : From left to right: Original, segmented sphere inside a cube (volume = 28%). The output of Algorithms 1 ($\mu > 0$), 1 ($\mu = 0$), 2, respectively, for volume = 50% of the cube's.