How to Escape a Declining Market: Capacity Investment or Exit?

Verena Hagspiel[a,*], Kuno J. M. Huisman[b,c], Peter M. Kort[b,d], Cláudia Nunes[e]

[a]*Department of Industrial Economics and Technology Management, Norwegian University of Science and Technology, 7491 Trondheim, Norway*
[b]*CentER, Department of Econometrics and Operations Research, Tilburg University, 5000LE Tilburg, The Netherlands*
[c]*ASML Netherlands B.V., Post Office Box 324, 5500 AH Veldhoven, The Netherlands*
[d]*Department of Economics, University of Antwerp, Belgium*
[e]*Department of Mathematics and CEMAT, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal*

Abstract

This paper considers a firm that faces a declining profit stream for its established product. The firm has the option to invest in a new technology with which it can produce an innovative product while having the option to exit at any point in time. In the presence of an exit option, earlier work determined the optimal timing to invest, where it was shown that higher uncertainty might accelerate investment timing.

In the present paper the firm also decides on capacity. This extension leads to monotonicity, i.e. higher uncertainty delays investment timing. We also find that higher potential profitability of the innovative product market increases the incentive to invest earlier, where, however, we get the counterintuitive result that the firm invests in smaller capacity. Finally, if quantity has a smaller negative effect on price, the firm wants to acquire a larger capacity, which in some situations results in an investment delay.

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*Corresponding author
Email address: `verena.hagspiel@iot.ntnu.no` (Verena Hagspiel)

1. Introduction

The photography industry underwent a disruptive change in technology during the 1990s when the traditional film was replaced by digital photography (see e.g. The Economist January 14th 2012). In particular Kodak was largely affected: by 1976 Kodak accounted for 90% of film and 85% of camera sales in America, making it the owner of a near-monopoly in America. While Kodak's revenues were nearly 16 billion in 1996, in 2011 it has decreased to 6 billion[1].

Kodak tried to get (squeeze) as much money out of the film business as possible and it prepared for the switch to digital film. The result was that Kodak did eventually build a profitable business out of digital cameras, but it lasted only a few years before camera phones overtook it. According to Mr. Komori, the former CEO of Fujifilm of 2000-2003, Kodak aimed to be a digital company, but that is a small business and not enough to support a big company. 'For Kodak it was like seeing a tsunami coming and there is nothing you can do about it', according to Mr. Christensen in The Economist (January 14th 2012).

This paper focuses on investment and exit decisions of a firm that has to deal with technological change. The above example showed that this can be a burden. However, there are enough examples of firms for which technological change brought fruitful times in terms of profits. One example is Activision, a successful company in the video game industry, where innovation plays a big role. Activision saw its worldwide sales increase with $650m in the first five days, when the new video game "Call of Duty: Black Ops" replaced its predecessor, "Call of Duty: Modern Warfare 2", in November 2010 (The Economist, December 10th 2011). Another example is the iPhone launched by Apple which was described by Time Magazine as 'the invention of the year 2007'. Apple's 2011 net income was $7.31bn in the three months up to June 25th, 125% higher than the previous year, making it the firm's record quarterly profit. Another

[1]See, for example, the Wall Street Journal on January 19, 2012 (http://blogs.wsj.com/deals/2012/01/19/kodak-bankruptcy-by-the-numbers/).

quarterly record was the revenue during that time period, a revenue of $28.6bn.

We study the problem of a price setting firm that produces with a current technology that faces a declining sales volume. The firm can either exit this industry or invest in a new technology with which it can produce an innovative product. The firm is a monopolist in a market characterized by uncertain demand, where the inverse demand function depends on a geometric Brownian motion process. Demand for the established product is characterized by a negative drift. Upon investment the firm is able to produce a new product, the demand of which is higher than demand of the established product. However, demand could still have a negative drift.

The question we study is when and if it is optimal to enter the innovative product market. In case the firm decides to launch the new product we also analyze the optimal capacity choice. Besides adopting the new technology, the firm also has the option to exit the market at any point in time. It can exit if it considers that the potential of the new product market is not profitable enough to invest and thus decides to exit instead of launching the new product. The exit option is conserved beyond the time of that potential investment in the new product. Therefore, the firm can also exit the market of the new product irrevocably at any time. Previous literature (Kwon (2010), Matomaki (2013)) considering the option to exit in combination with deciding about the optimal time to invest, found that it could be optimal to invest earlier when uncertainty goes up. We extend these papers by letting the firm also determine the optimal capacity size that should be acquired at the moment of investment, where the firm produces at capacity.

We derive the result that the optimal policy of the considered stopping problem exists and is unique. In addition we show that as uncertainty goes up, the firm invests in more capacity, which is an additional cause for investment delay. Unlike Kwon (2010) and Matomaki (2013), we find that this generates monotonicity regarding the effect of uncertainty on investment timing: when uncertainty goes up the firm invests later in a larger capacity level.

It turns out that innovative product market growth has a surprising effect in that the firm reduces investment size when the trend is higher. This is because timing is leading: a firm is eager to invest early in a fast growing market. Then the innovative output price is still low, which leads to a lower optimal capacity. An important characteristic of the new market is also in how strong output price is negatively affected by quantity sold. In fact this quantity is equal to the firm's capacity level because the firm produces at capacity. If this effect is larger the firm of course invests in a smaller capacity. Concerning timing things are not so clear. On the one hand, if quantity is strongly affecting price, the profitability of the new market is relatively low, driving the firm to invest later. On the other hand, in such a case the firm's optimal investment size is small, implying that investment costs are relatively low, which makes the firm decide to invest earlier.

This paper is organized as follows. We review related literature in Section 2. Our model is presented in Section 3, whereas Section 4 contains a benchmark model where the firm cannot exit. The comparative statics analysis of the optimal policies is conducted in Section 5. Our main results are presented in Section 6 and we conclude in Section 7. The appendix contains the proofs of all the propositions and a more in depth explanation of the calculus made in several parts of the paper.

2. Related Literature

A number of existing research contributions have analyzed several aspects of optimal technology adoption and exit decisions under uncertainty. There is extensive literature dealing with technology adoption (see Bridges et al. (1991) for an early review). Many papers formulated adoption decisions of new technology as stopping time problems. We refer to Hoppe (2002) for an extensive review of papers and Kwon (2010) for a review of more recent literature. We use a real options framework to model the technology investment decision.

Farzin et al. (1998) (see also Doraszelski (2001)) study the optimal timing of technology adoption when technology choice is irreversible and the firm faces a stochastic innovation process modeled by a compound Poisson process. Besides the uncertainty about the speed of the arrival the value of future improvements is assumed to be uncertain as well. They allow for multiple investments in new technology. Contrasting the optimal decision rule derived under the real options approach with that obtained under the net present value method, Farzin et al. (1998) show that the former implies a more cautious and slower pace of adoption than implied by the latter. This finding is in line with the conventional insight of real options literature about the effect of uncertainty on investment decisions: as uncertainty increases, it is optimal to wait longer before investment, reflecting the value of waiting (Dixit and Pindyck (1994)). In Farzin et al. (1998) the improvement of new technology follows a compound Poisson process. Recently, Hagspiel et al. (2015) extended Farzin et al. (1998) to a time-dependent intensity rate of new arrivals. They show that larger variance can accelerate investment in case the arrival rate rises while it can decelerate investment in case the arrival rate drops. Depending on whether the arrival rate is assumed to change or be constant over time, Hagspiel et al. (2015) show that the optimal technology adoption timing changes significantly.

Alvarez and Stenbacka (2001) characterize the optimal timing of when to adopt an incumbent technology, incorporating the opportunity to update this technology to future superior versions. In their study a switch of technology is assumed to generate a structural change in the cash flow, whereas the underlying stochastic process is assumed to be unchanged. They characterize how the real option values depend on market uncertainty and on the uncorrelated technological uncertainty regarding future new generations of technology. They show that in case the market uncertainty follows a geometric Brownian motion, an increase in uncertainty related to market as well as technological uncertainty delays optimal investment.

Some of the earliest work on entry and exit decisions goes back to Mossin (1968). McDonald and Siegel (1985), Brennan and Schwartz (1985) as well as Dixit (1989) are among the pioneering works that evaluate those decisions in the context of real options. McDonald and Siegel (1985) contemplate a case where operations can be suspended (mothballing decision), when operating profits are negative, and resumed at no additional costs if they turn positive again. Brennan and Schwartz (1985) introduce a model to optimally decide on opening, closing, and abandoning a mine. Dixit (1989) generalizes their framework assuming that there might be costs related to switching between suspension and an operating mode.

In our model the firm has the option to exit the market, which is considered to be an irreversible decision. This option to exit remains available also after investment. To our knowledge, there are only two papers that consider an exit option both before and after a possible investment. The first one to study this problem was Kwon (2010). Kwon (2010) analyzes the impact of uncertainty on a firm's optimal investment and exit decisions given that profit is expected to decline over time, in case the firm does not invest. The firm has the opportunity to make an investment that boosts the project's profit rate. He shows that it can be optimal to invest even in a declining market, and exit if the profit rate has deteriorated sufficiently.

(Matomaki, 2013, Article I) generalizes Kwon (2010), whose work relies on a Brownian motion with negative drift as underlying diffusion. He proves the existence and uniqueness of an optimal strategy when the stochastic process satisfies a general linear It$\hat{o}$ diffusion with different drifts and volatilities before and after the possible investment. Matomaki (2013) shows that for the case of a geometric Brownian motion with the same volatilities before and after investment (i.e. under the same assumptions as in this work), the effect of uncertainty on the investment threshold can be non-monotonic when the boost on the profit flow upon investment is relatively large. Specifically, the investment threshold first decreases and then increases in uncertainty.

We extend Kwon (2010) and Matomaki (2013) by also considering the size of the investment. This contrasts with the bulk of papers in the real options literature that only considers the time to invest. However, an investment decision is not only about timing but also about how much to invest. The papers that also take this into consideration, like Dixit (1993), Bar-Ilan and Strange

(1999), Dangl (1999), and Chronopoulos et al. (2013), mainly find that when uncertainty goes up the firm not only delays the time to invest, but also invests in a larger capacity. Hagspiel (2011) confirms this literature in that more uncertainty results in both delayed investment timing and and larger capacity. Recently, Huisman and Kort (2015) considered a framework where investing firms also have to deal with competition while determining their optimal capacity choice. We differ from this work by including the exit decision and a change in demand structure upon investment.

3. Model - Capacity, Timing

The firm currently operates in a declining market, producing an established product, denoted by index 1. The quantity is denoted by q_1, whilst the price is denoted by p_1. The relationship between the two is given by the following inverse demand function:

$$p_1(t) = \theta_1(t)(1 - \eta_1 q_1),$$

where η_1 is a positive constant, and the process $\theta_1 = \{\theta_1(t), t \geq 0\}$ follows a geometric Brownian motion, with dynamics

$$d\theta_1 = \alpha_1 \theta_1 dt + \sigma \theta_1 dz.$$

The stochastic process $\{\theta_1(t), t\}$, being proportional to the output price, represents how demand develops over time. It is governed by two parameters. The parameter $\alpha_1 (< 0)$ stands for the general market trend. Its negativity results in the declining market that we aim to model. How demand develops over time is uncertain beforehand. This part is captured by the term $\sigma\theta_1 dz$, so that the parameter σ represents the extent to which future demand is uncertain. By varying σ we can analyze how different levels of uncertainty affect dynamic firm behavior. z denotes a Brownian motion process. We assume that the firm is risk-neutral, and discounts against rate r, with $r > \alpha_1$. If this inequality does not hold, by choosing a later point to invest or exit the discounted revenue stream could be made indefinitely larger. Thus waiting longer would always be a better policy, and the optimum would not exist (see, e.g., Dixit and Pindyck (1994)).

The firm produces the established product with capacity K_1. It holds that $q_1 = K_1$, i.e. the firm always produces up to capacity. This assumption is often referred to as the 'market clearance assumption' (see, e.g., Goyal and Netessine (2007), Chod and Rudi (2005), Anand and Girotra (2007) and Deneckere et al. (1997)). Always producing up to capacity arise because firms may find it difficult to produce below capacity due to fixed costs associated with, for example, labor, commitments to suppliers, and production ramp-up (Goyal and Netessine (2007)). Even when firms can keep some capacity idle, a temporary suspension of production is often costly. This is the case, for example, due to maintenance costs needed to avoid deterioration of the equipment. Therefore, in practice firms often reduce prices to keep production lines running (see Goyal and Netessine (2007), Anand and Girotra (2007) and Mackintosh (2003)). However, counterexamples to the assumption of producing up to capacity also exist. Hagspiel et al. (2014) showed that allowing the firm to produce below capacity leads to larger capacity investment while the effect on timing shows a tradeoff: on the one hand the firm likes to invest earlier as the project is more valuable due to this volume flexibility, but on the other hand the firm has an incentive to invest later because investing in a larger capacity is more costly.

We distinguish between two types of cost. On the one hand the firm faces a fixed cost F. On the other hand it has to incur fixed unit production costs, which are equal to a constant c.

The firm has the option to start producing an innovative product, denoted by index 2, which requires an investment in production capacity. The capacity of the new product is denoted by K_2. The investment cost is a sunk cost and equal to δK_2, with δ being a positive constant.

Because this innovative market grows faster than the old one, we assume that the demand process change, and we denote by $\{\theta_2(t), t\}$ the demand over time for this innovative market, with

$$d\theta_2 = \alpha_2\theta_2 dt + \sigma\theta_2 dz, \tag{1}$$

where $\alpha_2 > \alpha_1$, since demand is higher for the new product. Still we impose that $\alpha_2 < r$ in order for a finite investment time to be optimal. Furthermore, if the firm decides to invest in this new market at time τ, then we assume that $\theta_2(0) = \theta_1(\tau)$, meaning that the process $\{\theta_2(t), t\}$ starts only to evolve after the firm decides to invest in the new market, and its initial value is precisely the demand level of the old market at the investment time.

Denoting the price and the quantity of the new product by $p_2(.)$ and q_2, respectively, at the moment of the new product launch the firm's demand function changes into:

$$p_2(t) = \theta_2(t)(1 - \eta_2 q_2), \tag{2}$$

where the constant η_2 is positive and such that $\eta_2 < \eta_1$. This inequality indicates we have vertical product differentiation in the sense that the new product is qualitatively better than the established one so that profit is larger. As in the first market, we assume that the firm produces up to capacity, i.e. $q_2 = K_2$.

The cost structure for the new product also changes after the new product launch. While the fixed cost still equals F, there are no variable costs. We motivate this assumption, on the one hand, by observing that in the digital world the unit cost of a product is most of the time very small. For many software products like for example video games, costs of producing an additional copy are very small or negligible. On the other hand our qualitative results carry over to a framework where unit costs are lower but positive for the second innovative product. Therefore, we set the unit cost of the second product to zero in order to save on notation.

Investing in the new product requires that the firm chooses the optimal time as well as the optimal size of the capacity investment. It can be the case that the new market is not profitable enough for an investment to be undertaken. Since the established product market is declining, it can be optimal for the firm to exercise the option to exit the market. We also allow for the possibility to exit the market after the investment in the innovative product has taken place. Therefore, the optimal stopping problem can be stated as follows, given that the current demand is θ:

$$\mathcal{V}(\theta) = \sup_{\tau_1} \mathbb{E}\left[\int_0^{\tau_1} e^{-rt}\Pi_1(\theta_1(t))dt + e^{-r\tau_1} \max\left\{0, \right.\right.$$
$$\max_{K_2}\left(\sup_{\tau_2 \mathbf{1}_{\{\tau_2>\tau_1\}}} \mathbb{E}\left[\int_{\tau_1}^{\tau_2} e^{-r(t-\tau_1)}\Pi_2(\theta_2(t-\tau_1), K_2)dt \middle| \theta_2(0) = \theta_1(\tau_1)\right]\right.$$
$$\left.\left.\left. -\delta K_2\right)\right\} \middle| \theta_1(0) = \theta\right], \tag{3}$$

meaning that until time τ_1 the firm is producing, earning a profit flow Π_1, that is a function of the demand process $(\theta_1(t))$. Time τ_1 is the first time that the firm decides either to invest in product 2 or to exit the market. If it is more profitable for the firm to exit the market, then the decision problem ends. If the firm decides to invest in the second market, then the instantaneous profit is Π_2, that depends on the new demand process $(\theta_2(t))$, until it decides to exit the market, which happens at time τ_2. Note that in equation (3) we make use of the assumption that the initial demand level in market 2 is equal to the demand level in market 1 when the firm decides to invest (i.e., $\theta_2(0) = \theta_1(\tau_1)$).

Finally, we note that τ_1 and τ_2 are both stopping times, and as the decision to exit market 2 has to be taken after deciding to invest in this market or either exiting, one must have $\tau_1 < \tau_2$ with probability one. Latter on, when we will present the solution to the problem, we will define τ_1 and τ_2 formally as stopping times.

To determine the value of investing in product 2, we first solve the subproblem that is stated at the right hand side of the maximization in equation (3). Considering a specific current value for $\theta_2(0) = \theta$ the net expected discounted profit of investing in product 2 is given by

$$\begin{aligned} V_2(\theta, K_2) &= \sup_{\tau_2} \mathbb{E}^\theta \left[\int_{\tau_1}^{\tau_2} e^{-r(t-\tau_1)} \Pi_2(\theta_2(t-\tau_1, K_2)) dt \right], \qquad (4) \\ &= \sup_{\tau_2} \mathbb{E}^\theta \left[\int_{0}^{\tau_2-\tau_1} e^{-rt} \Pi_2(\theta_2(t), K_2) dt \right], \\ &= \sup_{\tilde{\tau}} \mathbb{E}^\theta \left[\int_{0}^{\tilde{\tau}} e^{-rt} \Pi_2(\theta_2(t), K_2) dt \right], \end{aligned}$$

where $\mathbb{E}^\theta$ denotes the expectation with respect to the process θ_2, when its initial state is θ.

The optimal stopping problem in (4) is a standard problem. The instantaneous profit for product 2, when the current demand level is θ, is given by:

$$\Pi_2(\theta, K_2) = p_2 q_2 - F = \theta(1 - \eta_2 K_2) K_2 - F.$$

Stating the optimal stopping problem in the form of a Bellman equation and applying Ito's Lemma yields the following partial differential equation that $V_2(.)$ satisfies

$$\frac{1}{2}\sigma^2\theta^2 \frac{\partial^2 V_2(\theta, K_2)}{\partial \theta^2} + \alpha_2 \theta \frac{\partial V_2(\theta, K_2)}{\partial \theta} - r V_2(\theta, K_2) + \theta(1 - \eta_2 K_2) K_2 - F = 0.$$

Taking into account that there is an option to exit the market, standard calculations (see for example Dixit and Pindyck (1994)) lead to the following expression for the optimal value function V_2:

$$V_2(\theta, K_2) = \begin{cases} \frac{\theta K_2 (1-\eta_2 K_2)}{r-\alpha_2} - \frac{F}{r} + G\theta_2^{\beta_4}, & \theta > \theta_{E_2} \\ 0 & \theta \leq \theta_{E_2} \end{cases} \qquad (5)$$

where the exit threshold θ_{E_2}, as well as the specific expression of the unknown G, can be easily derived applying value matching and smooth pasting at the exit threshold:

$$\begin{aligned} V_2(\theta, K_2)\big|_{\theta=\theta_{E_2}} &= 0, \qquad (6) \\ \left.\frac{\partial V_2(\theta, K_2)}{\partial \theta}\right|_{\theta=\theta_{E_2}} &= 0. \qquad (7) \end{aligned}$$

Solving these equations one can easily derive the exit threshold θ_{E_2} and the expression for the parameter G:

$$\begin{aligned}\theta_{E_2} &= \left(\frac{\beta_4}{\beta_4 - 1}\right)\frac{F(r-\alpha_2)}{rK_2(1-\eta_2 K_2)},\\ G &= \theta_{E_2}^{-\beta_4}\left(\frac{1}{1-\beta_4}\right)\frac{F}{r}.\end{aligned}$$

Furthermore, β_4 is the negative root of the quadratic equation $\frac{1}{2}\sigma^2\beta(\beta - 1) + \alpha_2\beta - r = 0$. Moreover, we assume from now on that $r > \alpha_2$. We note that we use the notation $V_2(\theta, K_2)$ in order to emphasize the dependence of the value function not only on the actual demand level θ but also on the capacity of the new product K_2.

Next, consider the situation before the investment. Let us first determine the current instantaneous profits, and assume that the current demand is θ. The instantaneous profit equals:

$$\Pi_1(\theta) = \theta K_1(1 - \eta_1 K_1) - cK_1 - F.$$

Facing the declining profit stream in market 1, the firm has two possibilities, either to exit the market or to undergo investment to bring a new product on the market. We denote the exit threshold by θ_{E_1} and the investment threshold by θ_I, respectively. The following proposition states that the optimal policy for the stopping problem always exists and specifies the optimal value function of the firm.

Proposition 1 *The optimal policy for the stopping problem of equation (3) always exists. The optimal continuation region is $D^* = (\theta_{E_1}, \theta_I)$. It is optimal to exit the market when $\theta \leq \theta_{E_1}$ and invest in the new product when $\theta \geq \theta_I$. The corresponding value of the firm in the stopping region is equal to $\Omega(\theta) = \max\{0, V_2(\theta, K_2) - \delta K_2\}$.*

The optimal value function is uniquely given by

$$\mathcal{V}(\theta) = \begin{cases} V_1(\theta) & \text{for } \theta \in D^*,\\ \Omega(\theta) & \text{otherwise,}\end{cases}$$

where

$$V_1(\theta) = \frac{\theta K_1(1-\eta_1 K_1)}{r-\alpha_1} - \frac{cK_1+F}{r} + A_1\theta^{\beta_1} + A_2\theta^{\beta_2},$$

with A_1 and A_2 being constants to be derived such that $V(.)$ is continuous and differentiable at the boundary of D, and β_1 (β_2) is the positive (negative) root of the quadratic equation $\frac{1}{2}\sigma^2\beta(\beta-1)+\alpha_1\beta-r=0$.

Given that the firm invests in the new product, the optimal production capacity has to be determined. To do so, we compute $\max_{K_2\in[0,\infty]}\{V_2(\theta_I,K_2)-\delta K_2\}$. We can just compute the zero of $\frac{\partial(V_2(\theta_I,K_2)-\delta K_2)}{\partial K_2}$, and then check that $\frac{\partial^2(V_2(\theta_I,K_2)-\delta K_2)}{\partial^2 K_2}<0$.

To determine the optimal exit and investment policy we apply the following value matching and smooth pasting conditions:

$$\begin{aligned} V_1(\theta)|_{\theta=\theta_{E_1}} &= 0, &(8)\\ \left.\frac{\partial V_1(\theta)}{\partial\theta}\right|_{\theta=\theta_{E_1}} &= 0, &(9)\\ V_1(\theta)|_{\theta=\theta_I} &= V_2(\theta,K_2)|_{\theta=\theta_I}-\delta K_2, &(10)\\ \left.\frac{\partial V_1(\theta)}{\partial\theta}\right|_{\theta=\theta_I} &= \left.\frac{\partial V_2(\theta,K_2)}{\partial\theta}\right|_{\theta=\theta_I}. &(11)\end{aligned}$$

This leads to the results presented in Proposition 2.

Proposition 2 *The optimal capacity K_2 is implicitly given by the following equation:*

$$\frac{\theta_I(1-2\eta_2K_2)}{r-\alpha_2} - \delta + \frac{\partial G}{\partial K_2}\theta_I^{\beta_4} = 0, \tag{12}$$

with

$$\begin{aligned} \frac{\partial G}{\partial K_2} &= \left(\frac{\beta_4}{\beta_4-1}\right)\frac{F}{r}\theta_{E_2}^{-\beta_4-1}\frac{\partial\theta_{E_2}}{\partial K_2}, &(13)\\ \frac{\partial\theta_{E_2}}{\partial K_2} &= \left(\frac{\beta_4}{\beta_4-1}\right)\frac{F(r-\alpha_2)}{r}\left[\frac{\eta_2}{K_2(1-\eta_2K_2)^2}-\frac{1}{K_2^2(1-\eta_2K_2)}\right]. &(14)\end{aligned}$$

The investment and exit thresholds (θ_I and θ_{E_1}) are then solutions of the fol-

lowing equations:

$$\begin{aligned}
\frac{\theta_I K_1(1-\eta_1 K_1)}{r-\alpha_1} - \frac{\beta_1}{(\beta_1-1)}\frac{cK_1}{r} &+ \frac{(\beta_1-\beta_2)}{(\beta_1-1)}A_2(\theta_{E_1})\theta_I^{\beta_2} = \frac{\theta_I K_2(1-\eta_2 K_2)}{r-\alpha_2} \\
&- \frac{\beta_1}{(\beta_1-1)}\delta K_2 + \frac{(\beta_1-\beta_4)}{(\beta_1-1)}G\theta_I^{\beta_4}, \qquad (15) \\
\frac{\theta_I K_1(1-\eta_1 K_1)}{r-\alpha_1} - \frac{\beta_2}{(\beta_2-1)}\frac{cK_1}{r} &+ \frac{(\beta_2-\beta_1)}{(\beta_2-1)}A_1(\theta_{E_1})\theta_I^{\beta_1} = \frac{\theta_I K_2(1-\eta_2 K_2)}{r-\alpha_2} \\
&- \frac{\beta_2}{(\beta_2-1)}\delta K_2 + \frac{(\beta_2-\beta_4)}{(\beta_2-1)}G\theta_I^{\beta_4}. \qquad (16)
\end{aligned}$$

with A_1 and A_2 given by:

$$\begin{aligned}
A_1 &= \theta_{E_1}^{1-\beta_1}\left(\frac{1}{\beta_1-\beta_2}\right)\left[(\beta_2-1)\frac{K_1(1-\eta_1 K_1)}{r-\alpha_1} - \beta_2\theta_{E_1}^{-1}\left(\frac{cK_1+F}{r}\right)\right], \\
A_2 &= \theta_{E_1}^{1-\beta_2}\left(\frac{-1}{\beta_1-\beta_2}\right)\left[(\beta_1-1)\frac{K_1(1-\eta_1 K_1)}{r-\alpha_1} - \beta_1\theta_{E_1}^{-1}\left(\frac{cK_1+F}{r}\right)\right].
\end{aligned}$$

Finally, we are now in position to define formally the stopping times τ_1 and τ_2 used in Equation (3):

$$\tau_1 = \inf\{t : \theta_1(t) \notin (\theta_{E_1}, \theta_I)\}; \quad \tau_2 = \inf\{t : \theta_2(t) < \theta_{E_2}\} \qquad (17)$$

4. Benchmark Model

Allowing for the option to exit does not lead to explicit expressions for the relevant thresholds and the investment capacity level. For these reasons we consider a simplified model that is fully analytically tractable. This section presents such a model, which will serve as a benchmark for the analysis of the model introduced in Section 3. In particular, we set the variable and fixed production costs equal to zero, i.e., $F = c = 0$. The implication is that the firm has no incentive to exit. Analogous to Proposition 1, we have the following result.

Proposition 3 *The optimal policy for the stopping problem of equation (3) when $F = c = 0$, always exists. It is optimal to invest in the new product when $\theta \geq \theta_I$. The corresponding value of the firm in the stopping region is equal to $\Omega(\theta) = [V_2(\theta, K_2) - \delta K_2]$, with $V_2(\theta, K_2) = \frac{\theta K_2(1-\eta_2 K_2)}{r-\alpha_2}$. The optimal value* *function is uniquely given by*

$$\mathcal{V}(\theta) = \begin{cases} V_1(\theta) & \text{for } \theta < \theta_I, \\ \Omega(\theta) & \text{otherwise,} \end{cases}$$

where

$$V_1(\theta) = \frac{\theta K_1(1-\eta_1 K_1)}{r-\alpha_1} + A_1\theta^{\beta_1},$$

and β_1 *is the positive root of the quadratic equation* $\frac{1}{2}\sigma^2\beta(\beta-1)+\alpha_1\beta-r=0$.

Furthermore, we have the following value matching and smooth pasting conditions:

$$V_1(\theta) + A\theta^{\beta_1} = V_2(\theta, K_2) - \delta K_2|_{\theta=\theta_I}, \tag{18}$$

$$\frac{\partial V_1(\theta)}{\partial\theta} + \beta_1 A\theta^{\beta_1-1} = \frac{\partial V_2(\theta, K_2)}{\partial\theta} - \delta K_2\Big|_{\theta=\theta_I}, \tag{19}$$

From these conditions and maximizing the value function V_2 with respect to K_2 we obtain:

$$\theta_I(K_2) = \frac{\beta_1}{\beta_1-1}\frac{\delta K_2}{\frac{K_2(1-\eta_2K_2)}{r-\alpha_2} - \frac{K_1(1-\eta_1K_1)}{r-\alpha_1}}, \tag{20}$$

$$K_2(\theta) = \frac{1}{2\eta_2}\left(1 - \frac{\delta(r-\alpha_2)}{\theta}\right). \tag{21}$$

This leads to the investment rule presented in the following proposition.

Proposition 4 *The optimal capacity* K_2 *and the investment threshold* θ_I *are given by*

$$K_2 = \frac{1+\sqrt{1+\frac{4\eta_2(r-\alpha_2)(\beta_1^2-1)K_1(1-\eta_1K_1)}{r-\alpha_1}}}{2(\beta_1+1)\eta_2}, \tag{22}$$

$$\theta_I = \frac{(\beta_1+1)\delta(r-\alpha_2)}{\beta_1-\sqrt{1+\frac{4\eta_2(r-\alpha_2)(\beta_1^2-1)K_1(1-\eta_1K_1)}{r-\alpha_1}}}. \tag{23}$$

5. Comparative Statics

This section conducts a comparative statics analysis of the value of the firm after the investment on the innovative product, V_2, and the value of the firm for the whole situation, $\mathcal{V}$, respectively, as well as the exit threshold, θ_{E_2}, that relates to exiting after the firm has invested in the innovative product. We also present results concerning the probability of investment before exit, and the expected time to undertake a decision in the established market. The proofs of all propositions can be found in Appendix A.

We first establish the convexity of V_2, which is important for later comparative statics results.

Proposition 5 *The optimal return function V_2 is convex in θ.*

Next, we examine the comparative statics of V_2 with respect to α_2 and σ.

Proposition 6 *The optimal return function V_2 is non-decreasing in σ and strictly increasing in α_2.*

We employ these results to develop the comparative statics regarding the exit threshold in the innovative product market with respect to σ, stated in the following proposition.

Proposition 7 *If the optimal capacity in market 2, K_2, increases with the uncertainty then the exit threshold in market 2 (θ_{E_2}) decreases in σ.*

The non-increasing effect of uncertainty on the exit threshold can be explained as follows. When uncertainty is higher, the demand is more volatile. Hence, the firm is less convinced that after it exits, the demand will not pick up and increase again in the, possibly near, future. Therefore, the exit threshold is lower because the firm wants to keep the option alive for longer.

Regarding the analysis of the value of the firm, we again have to first establish its convexity in order to present the comparative statics of Proposition 9.

Proposition 8 *The optimal return function $\mathcal{V}$ is convex in θ.*

Proposition 9 *The value function of the firm $\mathcal{V}$ is increasing in σ and α_1, and strictly increasing in α_2.*

This result is intuitive: the value of the firm goes up if α_2 increases, since this implies that the output price grows faster after the firm has invested in the innovative technology. The value of the firm also increases in σ because upside potential is unlimited, while downside potential is limited by the output price being positive. The positive effect of α_1 on the value of the firm can also be justified: if θ's trend is larger, then the output price of the established product is expected to decrease at a lower pace.

Regarding the comparative statics of the investment and exit threshold of the established product market, we need to resort to numerical analysis. The insights resulting from this analysis are presented in Section 6. Another crucial result regards the point whether the firm will move on by investing in the innovative product or exiting. The next proposition provides an analytical result regarding the probability that the firm will eventually stay active by innovating.

Proposition 10 *The probability that the firm invests rather than exits, i.e. the probability that the threshold θ_I is hit before θ_{E_1}, is given by*

$$P_I = \frac{\left(\frac{\theta_0}{\theta_{E_1}}\right)^{1-\frac{2\alpha_1}{\sigma^2}} - 1}{\left(\frac{\theta_I}{\theta_{E_1}}\right)^{1-\frac{2\alpha_1}{\sigma^2}} - 1}.$$

When demand is governed by a Brownian motion with drift, like in Kwon (2010), then the expected time to undertake the decision is always infinite (as in this case the process is transient; see, for instance, Ross (1995)). In the case of a geometric Brownian motion, however, the mean exit time from an interval with compact closure is always finite (see, for instance, Lemma IV.2.1 in Bass (1998)). In the next proposition we provide expected values for relevant times.

Proposition 11 *The expected time until the firm decides to invest in the innovative product or exit the market (depending on which one occurs first) is given by:*

$$E[\tau_1] = \frac{1}{\frac{1}{2}\sigma^2 - \alpha_1}\left(\ln\left[\frac{\theta_0}{\theta_{E_1}}\right] - \frac{1-\left(\frac{\theta_0}{\theta_{E_1}}\right)^{1-2\frac{\alpha_1}{\sigma^2}}}{1-\left(\frac{\theta_I}{\theta_{E_1}}\right)^{1-2\frac{\alpha_1}{\sigma^2}}}\ln\left[\frac{\theta_I}{\theta_{E_1}}\right]\right),$$

if $\alpha_1 < \frac{1}{2}\sigma^2$*; otherwise it is infinite.*

If the firm decides to invest in the second market, then the expected time in the second market is equal to:

$$E\left[T_2\right] = \frac{\ln\left[\frac{\theta_I}{\theta_{E_2}}\right]}{\frac{1}{2}\sigma^2 - \alpha_2},$$

if $\alpha_2 < \frac{1}{2}\sigma^2$*; otherwise it is infinite. Furthermore, the expected time that the firm will stay in production is equal to*

$$\begin{aligned} E[\tau_2] = {} & \frac{\ln\left[\frac{\theta_I}{\theta_{E_2}}\right]}{\frac{1}{2}\sigma^2 - \alpha_2} \left(\frac{\left(\frac{\theta_0}{\theta_{E_1}}\right)^{1-\frac{2\alpha_1}{\sigma^2}} - 1}{\left(\frac{\theta_I}{\theta_{E_1}}\right)^{1-\frac{2\alpha_1}{\sigma^2}} - 1} \right) \\ & + \frac{1}{\frac{1}{2}\sigma^2 - \alpha_1} \left(\ln\left[\frac{\theta_0}{\theta_{E_1}}\right] - \frac{1 - \left(\frac{\theta_0}{\theta_{E_1}}\right)^{1-2\frac{\alpha_1}{\sigma^2}}}{1 - \left(\frac{\theta_I}{\theta_{E_1}}\right)^{1-2\frac{\alpha_1}{\sigma^2}}} \ln\left[\frac{\theta_I}{\theta_{E_1}}\right] \right). \end{aligned}$$

6. Results

This section studies effects of different parameters on the firm's investment decision. In our analysis we determined the effects for every parameter. However, in the following we choose to highlight the most important findings. To do so we start out analyzing the effect of uncertainty. Then we continue by establishing the effect of new market growth. Finally, we study the effect of the slope of the inverse demand curve.

6.1. Effect of uncertainty

The standard real options result says that the investment threshold goes up with increasing uncertainty reflecting the value of waiting. However, results can be different when realizing that upon investment the firm acquires an option to exit. In particular, Kwon (2010) obtains that when the profit boost upon investment is sufficiently large, volatility has a negative effect on the investment threshold. Furthermore, Matomaki (2013) finds that when the underlying process follows a geometric Brownian motion with changing drift upon investment, the effect of volatility on the investment threshold is non-monotonic. It decreases for relatively low values of uncertainty and then increases.

The reason that the relationship between uncertainty and investment threshold is ambiguous in the presence of an exit option, is that the value of the exit option increases with uncertainty. This exit option is acquired when investing. As a result the value of investment increases with uncertainty and therefore the firm may want to invest at a lower threshold level.

This section presents our findings regarding the effect of uncertainty on the investment decision. First of all we present the result for our benchmark model, where the absence of production costs implies that it is never optimal to exit. In this case Proposition 12 proves analytically that the usual result that the investment threshold goes up with uncertainty holds.

Proposition 12 *In absence of production costs (the benchmark model), the optimal capacity* K_2 *as well as the investment threshold* θ_I *are increasing in* σ.

This is not surprising because it is the presence of the exit threshold that may generate the opposite effect of a decreasing threshold under larger uncertainty. The proposition shows that in the benchmark model it also holds that increased uncertainty implies that the firm invests in a larger capacity. Intuitively, this is understandable, because when uncertainty goes up the typical aspect of asymmetric option valuation comes in: while the downward potential is limited by zero, the upward potential is unrestricted. In the later case the firm earns more because it is able to produce and sell a larger quantity. Note that investing in more capacity raises investment costs, which gives an additional incentive to delay the undertaking of the investment.

In our complete model, where the presence of (fixed) production costs makes that exit can be optimal at some point in time, we are not able to obtain analytical results concerning the effect of uncertainty. Tables 1 and 2, the former a case of a declining innovative market and the latter a case of an increasing innovative market, show that numerical results indicate that Proposition 12 carries

Table 1: Effect of increasing uncertainty on the optimal investment and exit strategy considering negative drift for the innovative product demand. (Parameter Values: $\alpha_1 = -0.02, \alpha_2 = -0.01, r = 0.1, c = 0.1, \eta_1 = 0.5, \eta_2 = 0.3, K_1 = 1.8, \delta = 10, F = 0.02$. In calculating $E[\tau_1]$ we choose $\theta_0 = \frac{(\theta_{E_1}+\theta_I)}{2}$.)

σ	0.05	0.1	0.15	0.2	0.25	0.3
K_2	0.410	0.572	0.735	0.874	0.985	1.079
θ_I	1.458	1.675	1.967	2.308	2.691	3.119
θ_{E_1}	1.057	0.947	0.827	0.714	0.614	0.526
θ_{E_2}	0.051	0.035	0.026	0.021	0.017	0.015
P_I	0.077	0.250	0.326	0.368	0.396	0.42
$E[T_{E_2}]$	298.04	257.09	203.58	156.65	121.43	97.04
$E[\tau_1]$	9.34	18.72	25.81	29.54	30.77	30.50
$E[\tau_2]$	32.28	83.10	92.14	87.21	78.84	70.82

Table 2: Effect of increasing uncertainty on the optimal investment and exit strategy considering negative drift for the innovative product demand. (Parameter Values: $\alpha_1 = -0.02, \alpha_2 = 0.01, r = 0.1, c = 0.1, \eta_1 = 0.5, \eta_2 = 0.3, K_1 = 1.8, \delta = 10, F = 0.02$. In calculating $E[\tau_1]$ we choose $\theta_0 = \frac{(\theta_{E_1}+\theta_I)}{2}$.)

σ	0.1	0.15	0.2	0.25	0.3
K_2	0.459	0.651	0.810	0.937	1.040
θ_I	1.242	1.477	1.752	2.057	2.392
θ_{E_1}	0.863	0.750	0.643	0.548	0.466
θ_{E_2}	0.038	0.026	0.020	0.016	0.013
P_I	0.328	0.359	0.384	0.405	0.421
$E[T_{E_2}]$	∞	3244.3	449.46	228.69	147.00
$E[\tau_1]$	12.70	20.41	25.16	27.33	27.82
$E[\tau_2]$	∞	1184.06	197.89	119.86	90.07

over to this case. This implies that the effect of having an option to exit, which may result in a lower investment threshold when uncertainty goes up, is not visible here. Apparently, the additional incentive to delay investment, caused by the fact that the firm wants to invest in a larger capacity in a more uncertain economic environment, dominates the negative effect on the investment threshold induced by the existence of the option to exit. We conclude that, despite the presence of the option to exit, allowing for capacity optimization restores the monotonic relationship between uncertainty and the investment threshold, which was lost in Kwon (2010) and Matomaki (2013).

Since in our model the uncertainty follows a geometric Brownian motion process, we can determine the effect of uncertainty on investment and exit timing (see the expressions for the expected time to invest or to exit in market 1, the expected time the firm spends in market 2, and the expected time that the firm is producing, in Proposition 11). The obtained results show that threshold effects do not directly translate in conclusions regarding timing. To see this, first note that the exit threshold θ_{E_2} is decreasing with uncertainty. However, Tables 1 and 2 show that the firm is expected to exit earlier in a more uncertain environment.

6.2. Effect of new market growth

In the benchmark model we can analytically prove the following result.

Proposition 13 *In absence of production costs (the benchmark model), the investment threshold, θ_I, and the optimal capacity in the second market, K_2, decrease with α_2.*

Hence, we obtain the at first sight surprising result that, when the potential profitability of the innovative product market is higher, it is optimal for the firm to invest in less capacity. The point is that the firm is more eager to invest in this case. This implies it will invest at a lower threshold level thus when the output price for the new product is still low. Therefore, it is optimal to invest in smaller capacity.

Table 3: Effect of increasing drift of the second market on the optimal investment and exit strategy. (Parameter Values: $\alpha_1 = -0.02, \sigma = 0.1, r = 0.1, c = 0.1, \eta_1 = 0.5, \eta_2 = 0.3, K_1 = 1.8, \delta = 10, F = 0.02$. In calculating $E[\tau_1]$ we choose $\theta_0 = \frac{(\theta_{E_1}+\theta_I)}{2}$.)

α_2	-0.02	-0.015	-0.01	0.00	0.01
K_2	0.629	0.602	0.5742	0.517	0.459
θ_I	1.928	1.800	1.678	1.450	1.242
θ_{E_1}	0.959	0.954	0.947	0.918	0.863
θ_{E_2}	0.034	0.035	0.035	0.037	0.038
P_I	0.211	0.230	0.250	0.291	0.328
$E[T_{E_2}]$	161.46	197.50	257.41	735.63	∞
$E[\tau_1]$	22.26	20.51	18.78	15.49	12.70
$E[\tau_2]$	56.36	65.90	83.07	229.56	∞

Numerical results, as presented in Table 3, indicate that this result carries over to the whole model, where the presence of costs makes exit worthwhile. This table also shows that, in case the firm has not invested yet, the exit threshold is lower when the expected growth in the new market is higher. This is because in the latter case the firm has a more profitable investment option, which also explains that the probability that the investment takes place increases.

Considering the exit threshold in the new market in Table 3, we conclude that it increases with α_2. The reason is that capacity decreases with α_2, which leads to a lower profit margin. To see this, note that instantaneous profit, $\theta(1 - \eta_2 K_2)K_2 - F$, admits its maximum value for $K_2 = \frac{1}{2\eta_2}$ (equalling 1.67 for the specific example in Table 3). If α_2 increases, K_2 decreases and in fact the instantaneous profit moves away from its maximum value. This gives an incentive to exit at a higher threshold. It may seem that the firm exits earlier when α_2 is higher. However, threshold values do not give perfect information about timing. This is exemplified by the fact that, when looking at the expected time to exit, we obtain the logical result that the firm is expected to exit later when new market growth is higher[2].

6.3. *Effect of the slope of the inverse demand curve*

Again we have an analytical result for the benchmark model without production costs.

Proposition 14 *In absence of production costs (the benchmark model), the investment threshold, θ_I, increases with η_2 and the optimal capacity in the second market, K_2, decreases with η_2.*

A larger value of η_2 means that quantity has a larger negative effect on the price in the new market. Therefore, the firm invests at a larger threshold value when η_2 is larger. Quantity having a larger negative effect on price implies that less capacity is needed. Hence, capacity decreases in η_2, despite the fact that the investment is undertaken at a larger threshold level.

Table 4: Effect of increasing η_2 on the optimal investment and exit strategy. (Parameter Values: $\alpha_1 = -0.02, \alpha_2 = -0.01, \sigma = 0.1, r = 0.1, c = 0.1, \eta_1 = 0.5, K_1 = 1.8, \delta = 10, F = 0.02$. In calculating $E[\tau_1]$ we choose $\theta_0 = \frac{(\theta_{E_1}+\theta_I)}{2}$.)

η_2	0.4	0.3	0.2	0.1	0.05
K_2	0.481	0.574	0.764	1.340	2.491
θ_I	1.788	1.678	1.584	1.502	1.465
θ_{E_1}	0.956	0.947	0.929	0.891	0.846
θ_{E_2}	0.043	0.035	0.026	0.014	0.008
P_I	0.232	0.250	0.263	0.267	0.258
$E[T_{E_2}]$	248.20	257.41	274.19	309.50	349.92
$E[\tau_1]$	20.28	18.78	17.69	17.38	18.12
$E[\tau_2]$	77.97	83.07	89.80	99.95	108.29

[2] In fact, the continuation region decreases with α_2 but the firm is expected to stay longer in that region.

Table 5: Effect of increasing η_2 on the optimal investment and exit strategy. (Parameter Values: $\alpha_1 = -0.02, \alpha_2 = 0.01, \sigma = 0.1, r = 0.1, c = 0.1, \eta_1 = 0.5, K_1 = 1.8, \delta = 10, F = 0.02$. In calculating $E[\tau_1]$ we choose $\theta_0 = \frac{(\theta_{E_1}+\theta_I)}{2}$.)

η_2	0.4	0.3	0.2	0.1	0.05
K_2	0.370	0.459	0.640	1.198	2.334
θ_I	1.278	1.242	1.210	1.183	1.174
θ_{E_1}	0.889	0.863	0.829	0.774	0.722
θ_{E_2}	0.055	0.038	0.027	0.014	0.007
P_I	0.328	0.328	0.322	0.304	0.280
$E[T_{E_2}]$	∞	∞	∞	∞	∞
$E[\tau_1]$	12.70	12.70	13.14	14.54	16.32
$E[\tau_2]$	∞	∞	∞	∞	∞

For the complete model we have numerical results that are presented in Table 4 for $\alpha_2 < 0$ and in Table 5 for $\alpha_2 > 0$. The result from the benchmark model does not completely carry over. Although Table 4 is in accordance with the proposition and in Table 5 we still have that capacity increases when η_2 becomes smaller, Table 5 also shows that the result of Proposition 14 is not valid anymore with respect to the investment threshold effects in case of production costs. In particular, we see that the investment threshold decreases with η_2 for small enough η_2. Apparently, capacity is leading here in the sense that an increased capacity makes that investment costs are higher so that it would be valuable to have a higher output price on the new market, and thus a higher investment threshold, at the moment of investment. Also, please note that comparing the first rows of Tables 4 and 5 shows that a larger trend in the new market results in a lower optimal capacity investment, which confirms the analysis in the previous section.

7. Conclusions

The paper studies a setting where a firm, currently operating in a declining established product market, has the option to invest and produce a new innovative product. We also include options to exit before and after the innovative investment.

The investment decision involves deciding about timing and about capacity size. In general the connection is that later timing implies investing in a market with higher output price, where it results in a larger optimal capacity size upon investment. We find that investment timing is leading where capacity size adjusts, in case we vary growth of the innovative product market. Consider a situation where the innovative product market is expected to grow faster. At first sight one would think this calls for a larger capacity. However, it turns out capacity will be smaller because the firm invests sooner in a market with larger growth.

With uncertainty it is the other way round. If capacity size is fixed, Kwon (2010) and Matomaki (2013) found that the effect of uncertainty on the investment threshold is ambiguous. We numerically show that introducing the capacity decision results in monotonicity where the investment threshold always goes up when the economic environment becomes more uncertain. This is caused by the fact that larger uncertainty calls for a larger capacity level, which gives an incentive for the firm to invest later, meaning a larger investment threshold. So here investment size is leading where investment timing adjusts.

Another characteristic important for the profitability of the new market is the extent to which quantity negatively influences price. If this effect is large, the profitability of the innovative market is low, and the firm invests later in less capacity. However, the fact that the size of the investment is low could also result in earlier investment. Then capacity is leading. We found that this could occur in an innovative product market with positive trend and relatively low effect of quantity on price.

This paper is one of the first fruits of a new research agenda that aims to enrich the real options literature, currently mainly focusing on investment timing, by determining also the investment size. Here it is important to assess which part of the decision is leading: timing or size. Another important topic is to include competition (an early contribution is Huisman and Kort (2015)).

Appendix A. Proofs

Appendix A.1. Proof of Proposition 1

Assume that $V_1(.)$ stated in the proposition is a candidate for the optimal value function. In the following we verify that $V_1(.)$ indeed satisfies all the sufficient conditions for being the optimal value function specified in Theorem 10.4.1 of Øksendal (2010). Oksendal's $\phi(.)$, $f(.)$ and $g(.)$ are here given by $V_1(.)$, $\Pi_1(.)$ and $V_2(\theta, K_2) - \delta K_2$, respectively. As $V = \Re_0^+$ and $D = (\theta_{E_1}, \theta_I)$, it holds that $\partial D = \{\theta_{E_1}, \theta_I\}$.

Conditions (iii), (iv), (viii) and (ix) hold trivially because θ follows a geometric Brownian motion. $V_1(.)$ is continuously differentiable in ∂D^* since we impose the value matching and smooth pasting conditions; Ω is also twice continuously differentiable $(\partial D^*)^c$, which proves that conditions (i) and (v), related with continuity, hold. Similarly, condition (ii) holds by definition of the value function V_1 (defined as the maximum).

Moreover, we introduce the following partial differential operator $L = \frac{\partial}{\partial t} + \alpha_1 \theta \frac{\partial}{\partial \theta} + \frac{1}{2}\sigma^2\theta^2\frac{\partial^2}{\partial \theta^2}$. Since the time-dependence of the return function is only through the discount factor e^{-rt}, the infinitesimal generator can be replaced by

$$L = -r + \alpha_1 \theta \frac{\partial}{\partial \theta} + \frac{1}{2}\sigma^2\theta^2\frac{\partial^2}{\partial \theta^2}.$$

To verify that (vi) and (vii) hold, we consider one case of V_1, while for the other cases similar calculations will apply:

$$V_1(\theta) = \frac{\theta K_1(1-\eta_1 K_1)}{r-\alpha_1} - \frac{cK_1 + F}{r} + A_1\theta^{\beta_1} + A_2\theta^{\beta_2}.$$

Therefore condition (vi) holds, since $LV_1(\theta) + \Pi_1(\theta) \leq 0$ on $\theta \in \Re_0^+ \setminus \bar{D}$. And condition (vii) holds because $LV_1(\theta) + \Pi_1(\theta) = 0$ for $\theta \in D$. □

Appendix A.2. Proof of Proposition 2

Follows automatically from the derivation in Section 3. $\square$

Appendix A.3. Proof of Proposition 3

See Proposition 1 for the case that $F = c = 0$. $\square$

Appendix A.4. Proof of Proposition 4

Standard calculations analogous to Proposition 2 lead to the result. $\square$

Appendix A.5. Proof of Proposition 5

By straightforward derivation, we get

$$V_2''(\theta) = \begin{cases} 0, & \theta \leq \theta_{E_2} \\ \beta_4(\beta_4 - 1)G\theta^{\beta_4 - 2}, & \theta > \theta_{E_2} \end{cases}. \tag{A.1}$$

Hence, as $G > 0$, the convexity follows.

Appendix A.6. Proof of Proposition 6

Let $\mu > 0$ and denote by $V_2(\theta, \alpha_2)$ the value function V_2 with the dependence on the drift of the process here denoted by α_2.

$$\begin{aligned} V_2(\theta, \alpha_2) =& \sup_{\tilde{\tau}} \mathbb{E}^{\theta_2(0)} \left[\int_0^{\tilde{\tau}} e^{-rt} \Pi_2 \left(\theta e^{\left(\left(\alpha_2 - \frac{\sigma^2}{2} \right) t + \sigma z_t \right)} \right) dt \right], \\ <& \sup_{\tilde{\tau}} \mathbb{E}^{\theta_2(0)} \left[\int_0^{\tilde{\tau}} e^{-rt} \Pi_2 \left(\theta e^{\left(\left(\alpha_2 + \mu - \frac{\sigma^2}{2} \right) t + \sigma z_t \right)} \right) dt \right], && \text{(A.2)} \\ \leq& V_2(\theta, \alpha_2 + \mu), && \text{(A.3)} \end{aligned}$$

where inequality (A.2) follows from the fact that $e^{\left(\left(\alpha_2 - \frac{\sigma^2}{2} \right) + \sigma z_t \right)} < e^{\left(\left(\alpha_2 + \mu - \frac{\sigma^2}{2} \right) + \sigma z_t \right)}$ with probability 1 and Π_2 is non-decreasing in θ. The inequality is strict since $\theta_2(0) > \theta_{E_2}$, and therefore $\tilde{\tau} > 0$. Moreover, as $\tilde{\tau}$ is the optimal stopping time for the problem with drift α_2 it is suboptimal for the problem with drift $\alpha_2 + \mu$, which proves inequality (A.3).

Concerning the non-decreasing behavior of V_2 as a function of the volatility σ, we refer to Ekstrom (2004), page 273, where in a note he refers that for convex contract functions the option price is non-decreasing in the volatility when the stock price follows a geometric Brownian motion. This result holds in our case as V_2 is convex in θ (see Proposition 5). $\square$

Appendix A.7. Proof of Proposition 7

In the following we present an auxiliary result that will be used for the proof of Proposition 7.

Lemma 1 *The negative root of the characteristic equation* $\frac{1}{2}\sigma^2\beta(\beta-1)+\alpha_2\beta-r=0$*, hereby denotes by* β_4*, increases with* σ*.*

Proof of Lemma 1

This result follows from Corollary 3 of Alvarez (2003).

$\square$

Proceeding with the proof of Proposition 7, we know that the exit threshold in market 2 is given by

$$\theta_{E_2} = \left(\frac{\beta_4}{\beta_4-1}\right)\frac{F(r-\alpha_2)}{r}\frac{1}{K_2(1-\eta_2 K_2)}. \tag{A.4}$$

We want to show the effect of increasing σ on θ_{E_2}. Therefore, we calculate the derivative of θ_{E_2} w.r.t σ:

$$\begin{aligned}\frac{d\theta_{E_2}}{d\sigma} &= \left[\left(\frac{1}{1-\beta_4}\right)\frac{F(r-\alpha_2)}{r}\frac{1}{K_2(1-\eta_2 K_2)}\right] \\ &\quad \left[\left(\frac{\beta_4'}{\beta_4-1}\right)+\beta_4\left(\frac{K_2'(1-2\eta_2 K_2)}{K_2(1-\eta_2 K_2)}\right)\right],\end{aligned} \tag{A.5}$$

where it is easy to see that the first part of the right hand side is positive. Therefore, the sign depends on the second term, which we denote by I, i.e.

$$I := \left(\frac{-\beta_4'}{1-\beta_4}\right)+\beta_4\left(\frac{K_2'(1-2\eta_2 K_2)}{K_2(1-\eta_2 K_2)}\right). \tag{A.6}$$

Given that $K_2 < \frac{1}{2\eta_2}$ (this holds because $K_2 = \frac{1}{2\eta_2}$ maximizes the revenue stream, so taking into account investment cost will result in a lower optimal K_2 level) and $\frac{\partial K_2}{\partial \sigma} > 0$, it holds that $I < 0$. This means that $\frac{d\theta_{E_2}}{d\sigma} < 0$.

$\square$

Appendix A.8. Proof of Proposition 8

First we note that the profit function, $\Pi_1(.)$, is a convex function in θ.

In a next step, we define

$$F(\theta_0) = \mathbb{E}\left[\int_0^{\tau} e^{-rt}\Pi_1\left(\theta_1(t)\right) dt \,\middle|\, \theta_1(0) = \theta_0 \right].$$

Then $F(\theta_0)$ is a convex function in θ_0 by the same reasoning that we used for equality (A.1).

Finally, taking into account that the maximization of V_2 over K_2 preserves the convexity of the function, as well as the maximum and the sum of two convex functions, then

$$\mathcal{V}(\theta_0) = \sup_{\tilde{\tau}} \mathbb{E}^{\theta_0}\left[\int_0^{\tilde{\tau}} e^{-rt}\Pi_1(\theta_1(t))dt + e^{-r\tilde{\tau}} \max\left\{0, \max_{K_2}\left(V_2(\theta_1(\tilde{\tau}), K_2) - \delta K_2\right)\right\}\right],$$

is also a convex function. $\square$

Appendix A.9. Proof of Proposition 9

By Proposition 6, V_2 is strictly increasing in α_2, and since α_2 affects only market 2, $\mathcal{V}$ is also strictly increasing in α_2.

Denote by $\mathcal{V}(\theta, \alpha_1)$ the value function $\mathcal{V}$ with the dependence on the drift of the process in market 1, here denoted by α_1. Let $\mu > 0$ (satisfying $\alpha_1 + \mu < \alpha_2$) and $\tilde{\tau}_{(\alpha_1)}$ be a stopping time adapted to the geometric Brownian motion with drift α_1. Then

$$\begin{aligned}
\mathcal{V}(\theta_0, \alpha_1) = {} & \sup_{\tilde{\tau}_{(\alpha_1)}} \mathbb{E}^{\theta_0}\left[\int_0^{\tilde{\tau}_{(\alpha_1)}} e^{-rt}\Pi_1\left(\theta_0 e^{\left(\left(\alpha_1 - \frac{\sigma^2}{2}\right)t + \sigma z_t\right)}\right) dt \right. \\
& \left. + \, e^{-r\tilde{\tau}_{(\alpha_1)}} \max\left\{0, \max_{K_2}\left(V_2(\theta(\tilde{\tau}_{(\alpha_1)}), \alpha_2) - \delta K_2\right)\right\}\right] && \text{(A.7)} \\
\leq {} & \sup_{\tilde{\tau}_{(\alpha_1)}} \mathbb{E}^{\theta_0}\left[\int_0^{\tilde{\tau}_{(\alpha_1)}} e^{-rt}\Pi_1\left(\theta_0 e^{\left(\left(\alpha_1 + \mu - \frac{\sigma^2}{2}\right)t + \sigma z_t\right)}\right) dt \right. \\
& \left. + \, e^{-r\tilde{\tau}_{(\alpha_1)}} \max\left\{0, \max_{K_2}\left(V_2(\theta(\tilde{\tau}_{(\alpha_1)})e^{\mu\tilde{\tau}_{(\alpha_1)}}, \alpha_2) - \delta K_2\right)\right\}\right] && \text{(A.8)} \\
\leq {} & \mathcal{V}(\theta_0, \alpha_1 + \mu), && \text{(A.9)}
\end{aligned}$$

where in (A.8) we use the fact that Π_1 is non-decreasing and that V_2 is non-decreasing in the initial value $\theta(\tilde{\tau}_{(\alpha_1)})$ because Π_2 is also non-decreasing in θ. Finally in (A.9) the sub-optimality of $\tilde{\tau}_{(\alpha_1)}$ is used.

In order to prove the behavior of $\mathcal{V}$ as a function of σ, we follow the "interesting consequence" of Theorem 4 of Alvarez (2003). Consider $d(x) = (r - \alpha_2)x$, which is an increasing function on $\Re^+$; then all the conditions of Theorem 4 of Alvarez (2003) are satisfied.

Let $\nu(\theta_0) = E^{\theta_0}\left[e^{-r\tilde{\tau}} f(\theta(\tilde{\tau}))\right]$, where in our case f denotes the return of stopping at $\tilde{\tau}$ (which includes the return of $\max\{0, V_2\}$). Since $\max\{0, V_2\}$ is non-decreasing in σ, as proved before, we conclude based on Theorem 4 of Alvarez (2003) that $\mathcal{V}$ is a non-decreasing function of the volatility parameter σ. $\square$

Appendix A.10. Proof of Proposition 10

Let $\theta(t)$ denote the demand level at time t, with $t < \tau_1$, so that the drift coefficient of the diffusion equation of θ is α_1. Therefore it follows that $\theta(t)$ is given by:

$$\theta(t) = \theta_0 \exp\left\{\left(\alpha_1 - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right\}.$$

We wish to have $\theta(t) > \theta_I$ before $\theta(t) < \theta_{E_1}$:

$$\theta(t) > \theta_I \Leftrightarrow z(t) > \frac{1}{\sigma}\ln\left(\frac{\theta_I}{\theta_0}\right) - \frac{1}{\sigma}\left(\alpha_1 - \frac{\sigma^2}{2}\right)t,$$
$$\theta(t) < \theta_{E_1} \Leftrightarrow z(t) < \frac{1}{\sigma}\ln\left(\frac{\theta_{E_1}}{\theta_0}\right) - \frac{1}{\sigma}\left(\alpha_1 - \frac{\sigma^2}{2}\right)t.$$

By Theorem 4.1 of Anderson (1960), we have that if $\{Y(t), t\}$ is a Wiener Process, if $\gamma_1 > 0, \gamma_2 < 0, \delta_1 = \delta_2 \neq 0$, then the probability that $Y(t) \geq \gamma_1 + \delta_1 t$ for a smaller t than any t for which $Y(t) \leq \gamma_2 + \delta_2 t$ is:

$$P_I = \frac{e^{-2\gamma_2\delta_1} - 1}{e^{2(\gamma_1-\gamma_2)\delta_1} - 1}.$$

For our case we have that

$$\gamma_1 = \frac{1}{\sigma}\ln\left(\frac{\theta_I}{\theta_0}\right) > 0,$$
$$\gamma_2 = \frac{1}{\sigma}\ln\left(\frac{\theta_{E_1}}{\theta_0}\right) < 0.$$
$$\delta_1 = -\frac{1}{\sigma}\left(\alpha_1 - \frac{\sigma^2}{2}\right).$$

Therefore:

$$e^{-2\gamma_2\delta_1} = \exp\left\{\ln\left(\frac{\theta_{E_1}}{\theta_0}\right)\left(2\frac{\alpha_1}{\sigma^2}-1\right)\right\} = \left(\frac{\theta_{E_1}}{\theta_0}\right)^{2\frac{\alpha_1}{\sigma^2}-1},$$
$$e^{2(\gamma_1-\gamma_2)\delta_1} = \exp\left\{-\ln\left(\frac{\theta_I}{\theta_0}/\frac{\theta_{E_1}}{\theta_0}\right)\left(2\frac{\alpha_1}{\sigma^2}-1\right)\right\} = \left(\frac{\theta_{E_1}}{\theta_I}\right)^{2\frac{\alpha_1}{\sigma^2}-1}.$$

So

$$P_I = \frac{\left(\frac{\theta_0}{\theta_{E_1}}\right)^{1-2\frac{\alpha_1}{\sigma^2}}-1}{\left(\frac{\theta_I}{\theta_{E_1}}\right)^{1-2\frac{\alpha_1}{\sigma^2}}-1}.$$

□

Appendix A.11. Proof of Proposition 11

In order to prove the expression for the expected time spent in market 1, that we denote by $E[\tau_1]$, we use the example in Section 10.9 of Wilmott (2006), with A given by α_1 and B given by σ^2, as the region Ω in our case is just the interval $[\theta_{E_1}, \theta_I]$ (a time homogeneous region).

The expected time in market 2, $E[T_2]$, is a standard result for the geometric Brownian motion (see, for instance, Ross (1995)).

Finally, as

$$\tau_2 = \tau_1 + \begin{cases} T_2 & \text{if the firm decided to invest in the second market,} \\ 0 & \text{if the firm decided to exit the first market ,} \end{cases}$$

then $E[\tau_2] = E[\tau_1] + E[T_2]P_I$. Therefore, the result follows from the probability of investment derived in the previous proposition and from the previous expressions regarding the expected times in the first market and in the second market.

□

Appendix A.12. Proof of Proposition 12

First, we will prove that the optimal capacity for the benchmark model as presented in equation (22) is increasing in the volatility parameter σ.

Denoting $a = \frac{4\eta_2(r-\alpha_2)K_1(1-\eta_1 K_1)}{r-\alpha_1}$ and $b = 2\eta_2$, we have

$$K_2 = K_2(\sigma) = \frac{1+\sqrt{1+a(\beta_1^2-1)}}{b(\beta_1+1)}, \tag{A.10}$$

where β_1 depends on σ. In order to ease the notation, we omit dependencies in σ in the following.

The derivative of K_2 with respect to σ is given by

$$\frac{\partial K_2}{\partial \sigma} = \frac{\left[1+a(\beta_1^2-1)\right]^{-\frac{1}{2}} a\beta_1\beta_1'(\beta_1+1) - \beta_1'(1+\sqrt{1+a(\beta_1^2-1)}}{(\beta_1+1)^2},$$

where β_1' is the short-hand notation for the first order derivative of β_1 (with respect to σ).

As the denominator of $\frac{\partial K_2}{\partial \sigma}$ is positive, we proceed to analyze the sign of the numerator, which we can re-write as follows

$$\begin{aligned} f(\sigma) &= a\beta_1'\beta_1\left[1+a(\beta_1^2-1)\right]^{-\frac{1}{2}}(\beta_1+1) - \beta_1'\left(1+\sqrt{1+a(\beta_1^2-1)}\right) \\ &= \beta_1'\left[\frac{a\beta_1(\beta_1+1)}{\sqrt{1+a(\beta_1^2-1)}} - \left(1+\sqrt{1+a(\beta_1^2-1)}\right)\right]. \end{aligned}$$

As $\beta_1' < 0$ (see (Dixit and Pindyck, 1994, Chapter 5)), it follows that $f(\sigma) \geq 0$ if and only if

$$\begin{aligned} \frac{a\beta_1(\beta_1+1)}{\sqrt{1+a(\beta_1^2-1)}} - \left(1+\sqrt{1+a(\beta_1^2-1)}\right) &< 0, \\ \frac{a\beta_1(\beta_1+1)}{\sqrt{1+a(\beta_1^2-1)}} - 1 - \sqrt{1+a(\beta_1^2-1)} &< 0, \\ a\beta_1(\beta_1+1) - \sqrt{1+a(\beta_1^2-1)} - (1+a(\beta_1^2-1)) &< 0, \\ (\beta_1+1)^2 a(a-1) &< 0. \end{aligned}$$

In order for this inequality to hold, it needs to hold that $a < 1$, which is equivalent to have

$$\frac{K_1(1-\eta_1 K_1)}{r-\alpha_1} < \frac{1}{4\eta_2(r-\alpha_2)}. \tag{A.11}$$

This condition has to hold in order for θ_I in equation (23) to be an admissible solution.

Using a similar procedure, we next prove that the investment threshold, θ_I, is also increasing in σ. The investment threshold can be stated as a function of K_2 as follows:

$$\theta_I(K_2) = \left(\frac{\beta_1}{\beta_1 - 1}\right)\left[\frac{\delta K_2}{\frac{K_2(1-\eta_2 K_2)}{r-\alpha_2} - \frac{K_1(1-\eta_1 K_1)}{r-\alpha_1}}\right].$$

Note that $\left(\frac{\beta_1}{\beta_1-1}\right)$ is increasing in σ (Dixit and Pindyck (1994)). Therefore, we just need to prove that the second part of the right hand side of equation (A.12) is also increasing in σ. Due to the previous result that $\frac{\partial K_2}{\partial \sigma} > 0$ holds, it remains to show that $h(K_2)$ is increasing in K_2. So let

$$h(K_2) = \frac{\delta K_2}{\frac{K_2(1-\eta_2 K_2)}{r-\alpha_2} - \frac{K_1(1-\eta_1 K_1)}{r-\alpha_1}}.$$

The derivative with respect to K_2 is equal to

$$\frac{\left(\frac{K_2(1-\eta_2 K_2)}{r-\alpha_2} - \frac{K_1(1-\eta_1 K_1)}{r-\alpha_1}\right) - K_2\left[\frac{(1-2\eta_2 K_2)}{r-\alpha_2}\right]}{\left(\frac{K_2(1-\eta_2 K_2)}{r-\alpha_2} - \frac{K_1(1-\eta_1 K_1)}{r-\alpha_1}\right)^2}. \tag{A.12}$$

As the denominator is always positive, $h(.)$ increases in K_2 if and only if:

$$\frac{\eta_2 K_2^2}{r-\alpha_2} - \frac{K_1(1-\eta_1 K_1)}{r-\alpha_1} \quad > \quad 0.$$

In the following we show that this condition always holds. To simplify notation we set $X = \frac{K_1(1-\eta_1 K_1)}{r-\alpha_1}$ and $Y = \frac{1}{4\eta_2(r-\alpha_2)}$. Note that $X < Y$ holds. Therewith, condition (A.13) can be rewritten as

$$1 + \sqrt{1 + \frac{(\beta_1^2 - 1)X}{Y}} \quad > \quad (\beta_1 + 1)\sqrt{\frac{X}{Y}}. \tag{A.13}$$

Straightforward calculations show that condition (A.13) is equivalent to $X < Y$. □

Appendix A.13. Proof of Propositions 13 and 14

Considering the expressions for K_2 and θ_I in equations (22) and (23), trivial computations show that both are decreasing in α_2.

It is clear from the expression of θ_I (equation (23)) that the investment threshold is increasing in η_2. With respect to the dependence of K_2 on η_2 we derive:

$$\frac{\partial K_2}{\partial \eta_2} = \frac{(\beta_1^2-1)K_1(r-\alpha_2)(1-\eta_1 K_1)}{(1+\beta_1)(r-\alpha_1)\eta_2\sqrt{1+\frac{4(\beta_1^2-1)(r-\alpha_2)\eta_2 K_1(1-\eta_1 K_1)}{(r-\alpha_1)}}} - \frac{1+\sqrt{1+\frac{4(\beta_1^2-1)(r-\alpha_2)\eta_2 K_1(1-K_1\eta_1)}{(r-\alpha_1)}}}{2\eta_2^2(1+\beta_1)}. \tag{A.14}$$

We want to show that $\frac{\partial K_2}{\partial \eta_2} < 0$. Multiplying $\frac{\partial K_2}{\partial \eta_2}$ by the strictly positive term

$$A = \sqrt{1+\frac{4(\beta_1^2-1)(r-\alpha_2)\eta_2 K_1(1-\eta_1 K_1)}{(r-\alpha_1)}},$$

gives

$$A\frac{\partial K_2}{\partial \eta_2} = -\frac{1+\frac{2(\beta_1^2-1)(r-\alpha_2)\eta_2 K_1(1-\eta_1 K_1)}{(r-\alpha_1)}+\sqrt{1+\frac{4(\beta_1^2-1)(r-\alpha_2)\eta_2 K_1(1-\eta_1 K_1)}{(r-\alpha_1)}}}{2(1+\beta_1)\eta_2^2}, \tag{A.15}$$

which is negative for the considered parameter ranges.

□

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